The Bonferroni Index of Income Inequality (*)

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Summary
This paper examines another old-new measure of income inequality: the Bonferroni index. Numerous definitions and interpretations are presented and discussed in the setting of several common models of income distribution. Various standard results are used to investigate the disaggregation of the index in terms of the factor components of total income, as well as, its decomposition by population subgroups and income classes.

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keywords: inequality measures, disaggregation of inequality measures, decomposition of inequality measures

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1. Introduction and Summary

C.E. Bonferroni (1930, p. 55 and p. 85) proposed a measure of income inequality, based on partial means, which is desirable when the major source of income inequality is the presence of units whose income is much below those of others. Although the index has been included in several Italian textbooks (e.g. Vajani 1974, pp. 196-200, Piccolo and Vitale 1981, p.90) the international literature did not absorb this measure so that it was neglected for many years. However, its inclusion in an important coverage of income distribution analysis (Nygard and Sandstrom, 1981) has attracted new attention on the Bonferroni index.

The aim of the present article is to review the theoretical and statistical properties of the Bonferroni index and to relate them to characteristics of the income distribution. The main results obtained can be summarized as follows:

a) The index concentrates on low incomes.
b) The index satisfies the diminishing transfer principle introduced by Mehran (1976).
c) The index is not additively decomposable.

The structure of the paper is as follows: the next section introduces the Bonferroni index and reviews its general properties. Section 3 examines the relationship between the index and inequality in the setting of several common models of income distribution. The disaggregation of the index by factor income components is studied in section 4, whereas, in section 5, is studied its decomposition by population subgroups.

2. The Bonferroni Index.

The scope of this section is to introduce the Bonferroni index both for continuous and discrete distributions of income and to discuss its general properties. It will be seen that the peculiarity of the index is the attribution of greater importance to transfers at the lower end of the income scale.

Definition of the Bonferroni index for continuous distributions.

Let *Y* be a nonnegative continuous random variable with cumulative distribution function *F(Y)*. The partial (or conditional) mean for *Y* over the interval *[0,y]* is given by

$$m(y) = \frac{\int_0^y u\,dF(u)}{F(y)} \tag{2.1}$$

For a given level *y* of the income *Y*

$$r(y) = \frac{\mu - m(y)}{\mu}; \qquad 0 < \mu < \infty \tag{2.2}$$

is a bounded, monotonic decreasing, and nonnegative function in *[0,∞)* and measures the relative difference between the total mean income μ and the mean of incomes less than or equal to *y*. Averaging (2.2) over all incomes yields the Bonferroni index

$$B=\int_0^\infty r(y)dF(y) \tag{2.3}$$

Note that, as suggested Pizzetti (1955) the Gini index *R* can be expressed as

$$R=\int_0^\infty r(y)\left[\frac{F(y)}{\int_0^\infty F(y)dF(y)}\right]dF(y) \tag{2.4}$$

hence, *R* is weighted mean of the *r(Y)*'s whereas *B* is their simple mean. Since *r'(Y)* and *F'(Y)* have opposite sign, then *B≥ R* (see De Vergottini, 1940).

The progressive mean *m(Y)* equals the ratio $\mu F_1(Y)/F(Y)$, where $F_1(Y)$ is the incomplete-first-moment distribution corresponding to F. Therefore, formula (2.3) can also be written as

$$B=\int_0^\infty\left[\frac{F(y)-F_1(y)}{F(y)}\right]dF(y)=1-\int_0^\infty\left[\frac{F_1(y)}{F(y)}\right]dF(y)=1-\int_0^\infty F_1(y)dLn[F(y)] \tag{2.5}$$

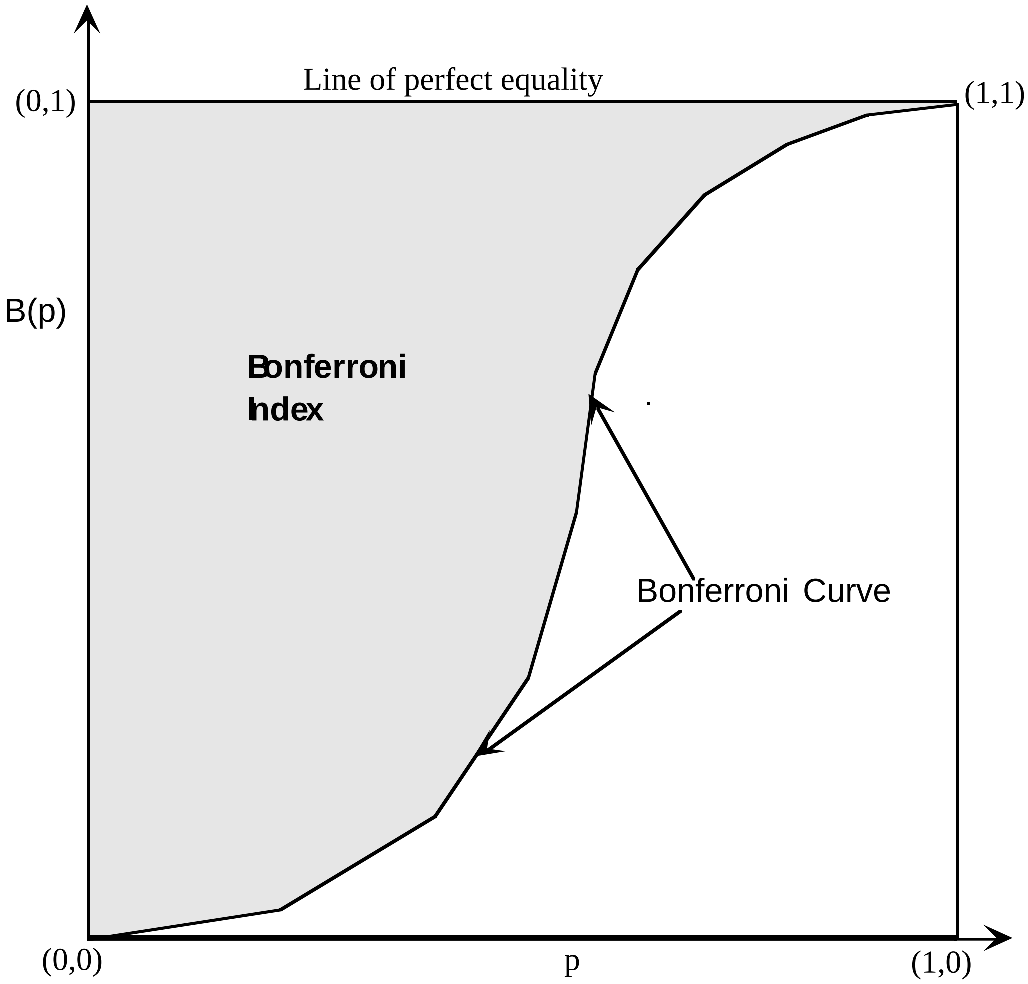
The Bonferroni index can also be regarded as the summary statistics of the Bonferroni curve of the income distribution. Such a curve, to be denoted *B(p):[0,1](⊃R)→[0,1],* is defined (see Zenga, 1984a, pp.70-77) as the relationship between the cumulative proportion *p=F(y)* of income receiving units (IRU's) and the ratio of the cumulative share of income *q=F(y)* and *p*. Let F^{-1}(t)= Inf{y| andF(t) =Inf be the inverse function of F(y) and F(y) respectively. Then

$$B(p)=\int_0^p\left[\frac{F_1^{-1}(t)}{F^{-1}(t)}\right]dt \tag{2.6}$$

The Bonferroni curve is represented in a unit square (figure 1). It is easy to check that *B(0)=0, B(1)=1*, and that *B(p)* is a nondecreasing function for *p∈[0,1]*. Clearly, perfect equality would result in points along the line *B(p)=1*, and if one IRU had all the income, the Bonferroni curve would coincide with the catheti OW and WZ.

From a geometric point of view, the Bonferroni index is the area between the Bonferroni curve and the line of perfect equality

$$B=1-\int_0^1 B(p)dp \tag{2.7}$$



Definition of the Bonferroni index for discrete distributions.
The population is assumed to consist of n IRU's who are labeled in nondescending order of income so that the subscript *i* indicates the rank of y_i among $y_1, y_2, \ldots, y_n$. Let μ be the arithmetic mean income, P_i be the cumulative population share and Q_i be the cumulative income share corresponding to the first i IRU's. Thus

$$P_i = \frac{i}{n}; \quad Q_i = \frac{\sum_{j=1}^{i} y_j}{n\mu} \qquad (i = 1, 2, \ldots, n) \tag{2.8}$$

The mean of incomes less than or equal to y_i is $M_i = \frac{1}{i} \sum_{j=1}^{i} y_j$; therefore, for a discrete distribution , we have

$$B_n = \frac{1}{n-1} \sum_{i=1}^{n-1} \left[\frac{\mu - M_i}{\mu} \right] = \frac{1}{n-1} \sum_{i=1}^{n-1} \left[\frac{P_i - Q_i}{P_i} \right] \tag{2.9}$$

these expressions show that B is readily estimable from existing data sources. Alternatively, B_n can be written as a ratio of linear combinations of order statistics

$$B_n = \frac{\sum_{i=1}^{n} w_i y_i}{\sum_{i=1}^{n} y_i} \tag{2.10}$$

with

$$w_i = 1 - \sum_{j=i}^{n} \frac{1}{j}; \quad w_{i+1} = w_i + \frac{1}{i}; \quad \sum_{i=1}^{n} w_i = 0 \tag{2.11}$$

w_i = 1 -; $w_{i+1} = w_i$ + ; = 0
(2.11)

This specification help characterize the income-weighting scheme in the welfare function behind the Bonferroni index. If a rank-preserving transfer of one unit of income takes place from the s-th to the r-th IRU (with $s>r$) B_n will change by an amount

$$\Delta B_n = -\frac{1}{(n-1)\mu} \sum_{j=r}^{s-1} \frac{1}{j} \tag{2.12}$$

which is proportional to the number of IRU's whose income falls in $[y_r, y_{s-1}]$. Furthermore, formula (2.12) shows that for a fixed difference $(s-r)$ between the two ranks, the lower is r the higher is ΔB_n so that the Bonferroni satisfies the diminishing transfer principle: a small positive transfer from a richer to a poorer unit decreases inequality and the decrease is larger the poorer the unit. It must be noted, however, that the effect of the transfer depends only on the ranks r and s and not on the size of income levels (see also Salvaterra, 1986).

De Vergottini (1940) interpreted (2.10) as the fraction of total income which should be transferred to achieve a perfect equality state by means of a gradual leveling of incomes starting from the poorest IRU. In fact, to equate y_1 and y_2, the quantity M_2 must be subtracted from y_2 and $(M_2 - M_1)$ added to y_1. Similarly, to eliminate the differences among $y_1=M_2$, $y_2=M_2$, and y_3, the quantity M_3 must be subtracted from y_3 and $(M_3 - M_2)$ added to each M_2. In general, $i(M_{i+1}-M_i)$ is transferred at the i-th redistribution so that the income transferred during the entire process of equalization is

$$\sum_{i=1}^{n-1} i\left(M_{i+1} - M_i\right) = \sum_{i=1}^{n-1} \left(\mu - M_i\right) \tag{2.13}$$

dividing (2.13) by the income that should be transferred in the case of complete inequality, that is $(n-1)\mu$, we obtain B_n.

It is easy to check that the Bonferroni index satisfies the following properties:

1. $0 \leq B \leq 1$.
2. $B = 0$ if and only if all incomes are equal.
3. $B = 1$ if and only if only one income is positive.
4. B is scale independent.
5. B is symmetric (does not depend on the assignment of labels to the IRU's)
6. Equal additions (subtractions) diminish (increase) B
7. B does not satisfy the property of invariance to the population replication (Zenga ,1986) has proved that a logical incompatibility exists between the third property and the property of invariance to the population replication)
8. B belongs to the class of linear measures of income inequality

$$J = \frac{\int_0^1 \left[F^{-1}(p) - \mu\right] W(p)dp}{\mu}, \quad \int_0^1 W(p)dp = 0 \tag{2.14}$$

defined by Mehran (1976) with *W(p)=1+Log(p)*. Mehran (1976) discussed the linear score function 2(p-1) corresponding to the Gini index and the quadratic score function W(p)=1-3(1-p) which does not correspond to a well-known inequality measure. "*Few other choice of W(p) seem to have been explored*" (Arnold, 1983, p.168). Both the authors failed to mention the Bonferroni index.

3. The Behavior of the Bonferroni Index in Some Income Models

In section 2 the analysis has been general, but to make further progress in relating the Bonferroni index and inequality it is necessary to assume a particular form of distribution. In this section, by using the technique defined by Zenga(1984), Pollastri(1986) and Dancelli (1986), I examine the behavior of the index when the distribution approximates the two extreme situations of minimum and maximum concentration. Table 1 provides the Bonferroni curve and the value of the Bonferroni index for some well known distribution functions.

The limiting situation for the rectangular model is, at one extreme, the perfect equality state which is approached for $\theta \to 0$. As θ rises the Bonferroni curve diverges from the line of equal distribution and for $\theta=1$ the curve becomes *L(p)=p* which is the other extreme compatible with the rectangular model. In general, all the rectangular models exhibit a rather low degree of inequality.

In the exponential model, as the minimum income θ goes to zero (while keeping constant the mean income), the Bonferroni curve will diverge from the equidistribution line and approach the most extreme Bonferroni curve possible: *L(p)=1+[(1-p)Log(1-p)]/p*. The same trend is realized when $\mu \to \infty$ and θ is fixed. Note that the exponential is another model whereby the absolute inequality state cannot be achieved.

As $\theta \to \mu$, that is, as the minimum income converges toward the mean income, (μ is still kept constant) the proportion of units receiving an income greater than μ tends to zero

Table 1: the Bonferroni curve and the Bonferroni index for some common distributions.

Model	*Distribution*	*Range*	*Bonferroni curve*	*Bonferroni index*
Rectangular	$F(y)=\frac{y-(1-\theta)\mu}{2\theta\mu}$	$y \geq \mu(1-\theta)$ $y \leq \mu(1+\theta)$ $0 \leq \theta \leq 1$	$B(p)=(1-\theta)+\theta p$	$\frac{\theta}{2}$
Exponential	$F(y)=1-e^{-\frac{y-\theta}{\mu-\theta}}$	$y \geq \theta, \;\; \theta > 0$	$B(p)=1+Ln\left[(1-p)^{\left(\frac{\mu}{\mu-\theta}\right)\left(\frac{1-p}{p}\right)}\right]$	$\frac{\mu-\theta}{\mu\gamma}$
Pareto/1	$F(y)=\frac{1}{\left[\left(\frac{\mu\sigma}{y}\right)\right]^{\frac{1}{1-\delta}}}$	$y>0; \sigma>0,$ $0<\delta \leq 1$	$B(p)=\frac{1-(1-p)^{\delta}}{p}$	$\psi(2)-\psi(1+\delta)$
Dagum/1	$F(y)=\frac{1}{\left[1+\left(\frac{\sigma}{y}\right)^{\frac{1}{\delta}}\right]^{\lambda}}$	$y>0; \lambda, \sigma>0,$ $0<\delta \leq 1$	$B(p)=\frac{IB\left(p^{\frac{1}{\lambda}}; \lambda+\delta, 1-\delta\right)}{p}$	$\frac{\left[\psi(\lambda+\delta)-\psi(\lambda)\right]}{\frac{1}{\lambda}}$

Note that μ indicates the total mean income for the first three models. Furthermore, γ ≅ 0.57721567 is the Euler's constant; IB(x;a,b) is the incomplete Beta function and ψ(x) is the Digamma function.

and the general distribution moves toward the state of perfect equality. For this model the Bonferroni is a linear function of θ; thus its values lead to unambiguous conclusions about the extent of concentration in the distribution. In particular, if $\theta=0$, then $B=\gamma$ and if $\theta=\mu$, then $B=0$.

For a distribution of incomes represented by a Pareto/I model the greatest departure from the condition of complete equality is obtained for δ∅0, the smallest for δ∅1 (the parameter δ is the well-known Gini delta). Because the Bonferroni is a one-to-one mapping of [0,1] onto itself, each value of *B* is the image of exactly one degree of inequality. In fact, it can easily be seen that if $\delta \to 1$ then $B \to 0$ and as the inequality increases ($\delta \to 0$), the value of B varies from zero to one.

To analyze the extreme cases of the distribution of incomes described by a Dagum/I model it is necessary to consider the role played by the two parameters λ and δ. First, suppose that λ is fixed. For $\delta \to 0$ the Bonferroni curve approaches the line of equidistribution; thus the degree of inequality tends to its minimum. As δ increases, the Bonferroni curve bends farther from $B(p)=1$ and for $\delta \to 1$ all the ordinates associated with a $p<1$ move toward zero; thus the extent of inequality tends to its maximum. When λ is fixed, the Bonferroni is a monotonically increasing function of δ: the larger the amount of inequality the greater the index; in particular, as it is easily derived from its expression in table 1, $B \to 0$ if $\delta \to 0$ and $B \to 1$ if $d \to 1$.

Suppose now that θ is given. For $\lambda \to 0$ the Bonferroni curve will approximate the equidistribution line and B converges to zero. On the other hand, for λ∅∞ the share of IRU's with income less than or equal to a given level decreases, that is the Bonferroni curve shifts towards the catheti of the maximum concentration triangle OWZ in figure 1 of section 2 and, accordingly, *B* converges to one (Note *that* $\psi(x) \sim Log(x)-0(1/x)$ for $x \to \infty$).

4. The Disaggregation of the Bonferroni Index by Factor Components of Income

The total income of an IRU is usually composed of income from several sources: wages and salaries, investment income, government transfer payments, etc. This section introduces an alternative expression for the Bonferroni index and shows that the Bonferroni for total income can be linked to the distributive shares and the factor Bonferronis; moreover, a useful categorization of the factor components is proposed.

Let the total income Y be the sum of G non negative factor income components

$$Y = \sum_{r=1}^{G} Y_r \tag{4.1}$$

The Bonferroni index for Y can be exactly decomposed into separate components for each type of factor income

$$B = \sum_{r=1}^{G} h_r w_r B_r \tag{4.2}$$

where μ_r, $h_r=\mu_r/\mu$ and B_r are, respectively, the mean income, the distributive share and the Bonferroni index for the r-th component. The w_r's are the weights (non necessarily positive) attributed to the components.

To prove (4.2) we express the Bonferroni index in terms of the covariance between the income variate Y and the logarithm of the cumulative distribution function $u_p=-Ln[F(Y)]$. Since $p=F(Y)$ is uniformly distributed between zero and one, u_p has a unit exponential distribution; in addition,

$$\int_0^1 \left(1-u_p\right)dp = 0$$

Therefore,

$$B = \int_0^1 -\frac{Y_p}{\mu}\left(u_p-1\right)dp - \int_0^1 -\mu\left(1-u_p\right)dp = \frac{1}{\mu}\int_0^1 \left(u_p-1\right)\left(\mu - Y_p\right)dp = Cov\left(Y;-Ln(F)\right) \tag{4.3}$$

by use of (4.3) equation (4.1) becomes

$$B = \frac{Cov\left(\sum_{r=1}^{G} Y_r, -Ln\left(F_r\right)\right)}{\mu} = \sum_{r=1}^{G} \frac{Cov\left(Y_r;Ln(F)\right)}{Cov\left(Y_r^{+};Ln\left(F_r\right)\right)} h_r B_r \tag{4.4}$$

where F_r is the cumulative distribution function of the r-th functional income source and

$$B_r = \frac{\sum_{r=1}^{G} Cov\left(Y_r^+; -Ln\left(F_r\right)\right)}{\mu_r} \tag{4.5}$$

is the factor Bonferroni index. In (4.5) Y_r indicates that the incomes from the r-th component are ranked in increasing order of magnitude, whereas Y_r indicates that those incomes are arranged in accordance with increasing magnitude of the total income. Since both -*Ln(F)* and -*Ln(*F_r*)* have the same variance, the ratio of covariances in (4.4) becomes

$$w_r = \frac{\rho\left(Y_r; Ln(F)\right)}{\rho\left(Y_r^+; Ln\left(F_r\right)\right)}; \quad r = 1, 2, \ldots, G \tag{4.6}$$

where ρ(x,y) denotes the simple correlation coefficient between x and y. For each factor, the so-called relative correlation coefficient w_r reflects the degree of concordance between the log-rank order of the IRU's by Y_r and their log-rank order by Y.

Equations (4.2) shows that the overall inequality depends on the degree of inequality within the distribution of each factor, on the importance of that factor in the total income, and on the amount of agreement between two sets of ordinal rankings. The Gini index is susceptible of an analogous decomposition with $w_r = \rho[Y;F]/\rho[Y_r;F_r]$ (Fields, 1979). The main difference between the two indices is that the Gini uses order statistics from the unit rectangular distribution whereas the Bonferroni uses order statistics from the unit exponential distribution.

Furthermore, the Bonferroni for the total income can be expressed as

$$B^* = \sum_{r=1}^{G} h_r B_r \tag{4.7}$$

i.e. as a weighted average (where the weights are the shares of the factor components) of all the factor Bonferronis if and only if the log-rank ordering of the IRU's is the same by every component, that is, if $w_r=1$, for $r=1,2, \ldots ,G$ It can also be shown (See Pyatt et al.1980) that unity is an upper bound for w_r and therefore the Bonferroni index cannot exceed B^*.

The w_r's provide a key for interpreting formula (4.2). The sign of a w_r depends only on the sign of $\rho[Y_r;Log(F)]$, hence a positive w_r implies that higher (lower) values of Y_r tend to be associated with higher (lower) values of *Y*; in this case the r-th factor is inequality-producing since the inequality of its distribution aggravates total inequality. Conversely, a negative w_r means that higher (lower) values of Y_r are prevalently associated with lower (higher) values of *Y*; therefore, the factor Y_r is inequality-reducing since its distribution alleviates total inequality. A component for which $w_r=0$ can be considered neutral since the total inequality would remain the same either if it were included in, or if it were excluded from, the total income.

Numerical Illustration

Table 2 reports the individual data of a ten-unit society whose members receive income from three different sources: W, X, Z.

Table 2: income distribution by income sources

Unit	W	X	Z	Total
1	2	100	100	202
2	4	100	90	194
3	8	100	80	188
4	16	100	70	186
5	32	100	60	192
6	64	100	50	214
7	128	100	40	268
8	256	100	30	386
9	512	100	20	632
10	1024	100	10	1134
Sum	1790	1000	550	3340

Applying the source disaggregation methodology to the data of table 2, we obtain the decomposition statistics given in table 3.

Table 3: decomposition of inequality for data in table 2.

Source	Mean	Factor Income Share	Relative Correlation	Bonferroni
W	204.6	0.569	0.9740	0.8570
X	100.0	0.278	0.0000	0.0000
Z	55.0	0.153	-0.6680	0.4550
Total	359.6	1.000		0.4290

We see from the factor Bonferronis that income from *Z* is moderately concentrated, that income from *W* is highly unequally distributed, and that all the units receive an equal amount of factor income *X* (of course, source *X* is a neutral component with respect of the total income). However, since source W accounts for more than a half of the total income, the overall Bonferroni is dominated by the value of 0.857 found for this component. The relative correlation coefficients reveal that the log-ranking of the units by income from W is strongly correlated with their log-ranking by total income. Conversely, an inverse relationship (w_r=-0.668) exists between factor income *Z* and total income, so that total inequality is reduced.

5. The Decomposition of the Bonferroni Index

This section investigates the possibility of decomposition of the Bonferroni index when the IRU's are cross-classified according to k mutually exclusively income classes $\{[Y_{i-1},Y_i),Y_{i-1}<Y_i;\ i=1,2,...,k)$ and G disjoint population subgroups $\{S_1,S_2,...,S_G\}$. It will be shown that the Bonferroni index, like the Gini index (Bottiroli Civardi, 1985), is not an additively decomposable measure

Table 4 below is a *(k x G)* contingency table for the comparison of the *G* population subgroups with respect to the k classes of income

Table 4: cross-classification of IRU's

Classes	S_1	S_2	S_r	S_G	
$[Y_0,Y_1)$	(μ_{11},n_{11})	(μ_{12},n_{12})	(μ_{1r},n_{1r})	(μ_{1G},n_{1G})	(μ_1^+,n_1^+)
$[Y_1,Y_2)$	(μ_{21},n_{21})	(μ_{22},n_{22})	(μ_{2r},n_{2r})	(μ_{2G},n_{2G})	(μ_2^+,n_2^+)
$[Y_{i-1},Y_i)$	(μ_{i1},n_{i1})	(μ_{i2},n_{i2})	(μ_{ir},n_{ir})	(μ_{iG},n_{iG})	(μ_G^+,n_G^+)
$[Y_{k-1},Y_k)$	(μ_{ik},n_{k1})	(μ_{k2},n_{k2})	(μ_{kr},n_{kr})	(μ_{kG},n_{kG})	(μ_k^+,n_k^+)
	(μ_1,n_1)	(μ_2,n_2)	(μ_r,n_r)	(μ_G,n_G)	(μ,n)

The couples in this table indicate

(μ_{ir},n_{ir}) The mean income for the i-th class of the r-th population subgroup (it is assumed that income is uniformly distributed within each income class) and the number of units receiving μ_{ir}.

(μ_r,n_r) The mean income and the size of the r-th subgroup.
(μ_i,n_i) The mean income and the size of the i-th class.
(μ,n) The mean income and the size of the whole population.

The following notations are used in the formulae for the decomposition of the Bonferroni index

$N_{ir} = \sum_{j=1}^{i} n_{jr}$ the cumulative sum of units in subgroup S_r

$F_{ir}=N_{ir}/n_r$ the cumulative proportion corresponding to N_{ir}

$M_{ir} = \frac{\sum_{j=1}^{i} \mu_{jr} n_{jr}}{N_{ir}}$ the progressive mean income for units located in S_r

$f_r = (n_r/n)$ relative frequency of S_r

$h_r = \mu_r f_r/\mu$ the share of income attributable to S_r.

$N_i = \sum_{j=1}^{i} n_j^+$ cumulative sum of units whose income is less than Y_i

$F_i = N_i / n$ cumulative proportion corresponding to N_i

$M_i = \frac{\sum_{r=1}^{G} M_{ir} N_{ir}}{N_i}$ the progressive mean for the whole population

$B_r = \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{(\mu_r - M_{ir})}{\mu_r}$ the Bonferroni index for the r-th subgroup

The Bonferroni index for the distribution of income of the whole population can be decomposed as follows

$$B = B^w + B^a \qquad (5.1)$$

where

$$B^w = \sum_{r=1}^{G} h_r B_r; \qquad B^a = \sum_{r=1}^{G} h_r \left(1 - \frac{F_{ir}}{F_i}\right) \frac{M_{ir}}{\mu_r} \qquad (5.2)$$

so that the Bonferroni index is expressed as the sum of a Within-group component (a weighted average of the partial Bonferronis where the weights are equal to the subgroup income shares) and a weighted average, the Across component, of other subgroup characteristics. In this way, equation (5.1) tell us the relative importance of inequality within subgroups as compared with diversity in mean incomes across subgroups.

To understand the nature of the term B^a, suppose that there is complete association (from another point of view this means that the incomes in the r-th group completely dominate the incomes in the (r+1)-th group for *r=1,2,..,G-1)* between classes and subgroups: each IRU receives the mean income of its subgroup (for each column of table 4, only one cell has a nonzero frequency). Then we have:

$$M_{ir} = \mu_r \qquad \{(i=1,2, \ldots ,k); (r=1,2, \ldots ,G)\} \qquad (5.3)$$

and, as a consequence

$$B = \sum_{r=1}^{G} h_r \left[\frac{1}{k-1} \left(1 - \frac{F_{ir}}{F_i}\right) \right] = B^b \qquad (5.4)$$

in other words, B^b is the Between-group inequality component of the Bonferroni index, i.e. the inequality that would remain if Within group inequality were eliminated.

Suppose now that the frequencies of table 4 satisfy the condition of independence: the place of a unit in one subgroup is irrelevant to its position on the income ladder. This condition implies that

$$n_{ir} = \frac{n_r n_r^+}{n} \qquad \{(i=1,2, \dots ,k); (r=1,2, \dots ,G)\} \tag{5.5}$$

It immediately follows that $N_{ir}=n_i f_r$ and, a fortiori, that $(F_{ir}/F_i) =1$; that is, all the conditional distributions are equal to the total distribution and therefore, $B=B^w$; hence, the aggregate Bonferroni is a weighted average (weights summing to one) of the subgroup Bonferronis only if the subgroup distributions are identical. As the frequencies of table 4 depart from independence, B differs from B^w. In general, the Across term B^a can be written as $B^a = B^b + B^i$ where

$$B^i = \frac{1}{k-1}\sum_{i=1}^{k-1}\sum_{r=1}^{G} h_r\left(1-\frac{M_{ir}}{\mu_r}\right)\left(\frac{F_{ir}}{F_i}-1\right) \tag{5.6}$$

is an Interaction term whose value depends on the extent by which the rank of an IRU is affected by the group to which it belongs. Mehran (1975) interpreted the interaction term as a measure of the extent of income domination in one group over the others apart from the difference between their mean incomes. See also Ferrari and Rigo (1987)

The complete decomposition of the Bonferroni index is finally given by

$$B = B^w + B^i + B^b \tag{5.7}$$

Because of the factor (F_{ir}/F_i) in B^a the knowledge of the aggregate characteristics of the subgroups $(B_r, f_r, \mu_r; r=1,2, \dots ,G)$ does not suffice to compute B. In this sense B is not aggregative (Bourguignon, 1979) and hence the Bonferroni index, like the Gini index, is not an additive decomposable inequality measure. This result is consistent with the fact that the Bonferroni index does not belong to the class

$$I(\alpha) = \frac{1}{\alpha(\alpha-1)}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i}{\mu}\right)^{\alpha}-1\right\} \tag{5.8}$$

of the additively decomposable inequality measures studied by Shorrocks (1980) and Zenga (1986).

The implications of the decomposition methodology studied in this section can be brought out most clearly with a numerical illustration. In this illustration we deal with two sectors, say Urban (U) and Rural (R). The distribution in the latter is of moderate inequality: $B_R=0.2753$ whereas the distribution of the former is much more unequal: $B_U=0.5103$.

Table 5: distribution of income by income classes and geographical areas.

Classes		Urban Sector		Rural Sector	
Y_{i-1}	Y_i	μ_{iU}	f_{iU}	μ_{iR}	f_{iR}
[0	10)	5	0.06	9	0.45
[10	20)	15	0.16	18	0.30
[20	30)	25	0.26	26	0.15
[30	50)	40	0.22	44	0.05
[50	70)	60	0.20	65	0.03
[70	110)	90	0.10	100	0.02

We assume that the per capita income in the urban sector $\mu_U=39$, is twice as large as per capita income in the rural sector: $\mu_R=19.5$. Finally, we suppose that the proportion of the IRU's in sector U increases from 20% to 80% in such a way as to leave unaltered the distribution of income within each sector.

Table 6: decomposition statistics for the Bonferroni index.

f_U	h_U	μ	B	B^w	B^i	B^b	B^a
0.2	0.3333	23.4000	0.3559	0.3536	-0.0710	0.0732	0.0023
0.3	0.4615	25.3500	0.3860	0.3838	-0.0941	0.0963	0.0022
0.4	0.5714	27.3000	0.4112	0.4096	-0.1113	0.1119	0.0006
0.5	0.6667	29.2500	0.4325	0.4320	-0.1201	0.1206	0.0005
0.6	0.7500	31.2000	0.4508	0.4516	-0.1232	0.1224	-0.0008
0.7	0.8235	33.1500	0.4664	0.4688	-0.1186	0.1162	-0.0024
0.8	0.8889	35.1000	0.4805	0.4842	-0.1036	0.0999	-0.0037

As the proportion of IRU's in sector R declines both the mean income and the Bonferroni index for the whole population decreases. The term Between rises at first and later falls, while the Interaction term has an opposite behavior: falls at first an then rises. It must be particularly stressed the balance effect between the term B^i and B^b which reduces the component across B^a almost to the vanishing point and make it possible to approximate accurately the total Bonferroni by its within component.

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