1 Profit maximization

Behavioral assumption:

$$\max R(a_1, a_2, ...a_n) - C(a_1, a_2, ...a_n)$$

an optimal set of actions

$$\mathbf{a}^* = (\mathbf{a}_1^*, ...\mathbf{a}_n^*)$$

is characterized by the conditions:

$$\frac{\partial R(\mathbf{a}^*)}{\partial a_i} = \frac{\partial C(\mathbf{a}^*)}{\partial a_i}$$

The firm's profit maximization problem reduces to choice of the price of outputs (or price of inputs) and levels of output (or inputs).

The firm faces:

- Technological Constraints: feasibility of the production plan.
- Market constraints: effect of actions of other agents on the firm.

For competitive firms we assume *price taking* behavior.

1.1 Description of technology

Suppose the firm has n possible goods to serve as inputs and/or outputs. We represent a specific production plan by a vector $\mathbf{y}$ in R^n where y_i is negative if the i^{th} good serve as a net input and positive if it serves as a net output. Such a vector is called a *net output* or *netput* vector. The set of all feasible production plans (netput vectors) is called *production possibilities* set (Y a subset of R^n).

In the short run: Restricted or short-run production possibilities $Y(z)$ consists of all feasible netput bundles compatible with the constraint z.

1. A firm produces one output with vector of inputs $\mathbf{x}$.

$$V(y) = \left\{\mathbf{x} \text{ in } R^n_+\text{: } (y, ... - \mathbf{x}) \text{ is in } Y\right\}$$

2. For the case above we can define the isoquant:

$$Q(y) = \left\{\mathbf{x} \text{ in } R^n_+\text{: } \mathbf{x} \text{ is in } V(y), \mathbf{x} \text{ is not in } V(y') \text{ for } y' > y\right\}$$

the isoquant gives all input bundles that produce exactly y.

3. If the firm has only one output we can define the production function:

$$f(\mathbf{x}) = \{y \text{ in } R\text{: } y \text{ is the maximum output associated with} - \mathbf{x} \text{ in } Y\}$$

4. A production plan $\mathbf{y}$ in Y is called efficient if there is no such $\mathbf{y}'$ such that $\mathbf{y}' \geqq \mathbf{y}$, i.e. if there is no way to produce more output with the same inputs or to produce the same output with less inputs. We describe the set of efficient production plans by some function $T{:}R^n \to R$ where $T(\mathbf{y}) = 0$ only if $\mathbf{y}$ is efficient.

- Example 1: Cobb-Douglas

$$Y = \left\{(y, -x_1, -x_2) \text{ in } R^3\!: \; y \;\leqq x_1^{\alpha} x_2^{1-\alpha}\right\} \quad 0 < \alpha < 1$$

$$V(y) = \left\{(x_1, x_2) \text{ in } R^2_+ {:} y \;\leqq x_1^{\alpha} x_2^{1-\alpha}\right\}$$

$$Q(y) = \left\{(x_1, x_2) \text{ in } R^2_+ {:} y = x_1^{\alpha} x_2^{1-\alpha}\right\}$$

$$Y(z) = \left\{(y, -x_1, -x_2) \text{ in } R^3\!: \; y \;\leqq x_1^{\alpha} x_2^{1-\alpha},\, x_2 = z\right\}$$

$$T(y, x_1, x_2) = y - x_1^{\alpha} x_2^{1-\alpha}$$

$$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

- Leontief technology

$$Y = \left\{(y, -x_1, -x_2) \text{ in } R^3\!: \; y \leqq \min\left(ax_1, bx_2\right)\right\}$$

$$V(y) = (x_1, x_2) \text{ in } R^2_+\!: \; y \leqq \min\left(ax_1, bx_2\right)$$

$$Q(y) = \left\{(x_1, x_2) \text{ in } R^2_+ {:} y = \min\left(ax_1, bx_2\right)\right\}$$

$$T(y, x_1, x_2) = y - \min\left(ax_1, bx_2\right)$$

$$f(x_1, x_2) = \min\left(ax_1, bx_2\right)$$

1.2 Description of Production Sets and Input Requirement Sets

The input set is monotonic if:

Monotonicity:

$$\mathbf{x} \text{ is in } V(y) \text{ and } \mathbf{x}' \geqq \mathbf{x}, \text{ then } \mathbf{x}' \text{ is in } V(y)$$

The input set is convex if:

Convexity:

if $\mathbf{x}$ is in $V(y)$ and $\mathbf{x}'$ is in $V(y)$ then $t\mathbf{x} + (1-t)\mathbf{x}'$ is in $V(y)$ for all $0 \leqq t \leqq 1$. That is $V(y)$ is a convex set.

1.3 The technical rate of substitution

Consider $x_n(x_1,...x_{n-1})$ the (implicit) function that tell us how much of x_n it takes to produce y if we are using $x_1,...x_{n-1}$ of the other factors. Then the function $x_n(x_1,...x_{n-1})$ has to satisfy the identity:

$$f(x_1, ...x_{n-1}, x_n(x_1, ...x_{n-1}) \equiv y$$

We are interested for an expression for $\partial x_n(x_1^*,...x_{n-1}^*)/\partial x_1$. Differentiating the above identity we find:

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_1} = \frac{\partial f(\mathbf{x}^*)}{\partial x_n}\frac{\partial x_n(x_1^*, ...x_{n-1}^*)}{\partial x_1} = 0$$

or

$$\frac{\partial x_n(x_1^*, ...x_{n-1}^*)}{\partial x_1} = -\frac{\partial f(\mathbf{x}^*)/\partial x_1}{\partial f(\mathbf{x}^*)/\partial x_n}$$

Example: Technical rate of Substitution in a Cobb-Douglas Technology. $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$

$$\frac{\partial f(x)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$$

$$\frac{\partial f(x)}{\partial x2} = (1-\alpha) x_1^\alpha x_2^{-\alpha}$$

$$\frac{\partial x_2(x_1)}{\partial x_1} = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^\alpha x_2^{-\alpha}} = -\frac{\alpha}{(1-\alpha)}\frac{x_2}{x_1}$$

1.4 Returns to Scale

Constant Return to Scale. A technology exhibits constant return to scale if any of the following are satisfied:

1. $\mathbf{y}$ is in Y implies $t\mathbf{y}$ is in Y for all $t > 0$.
2. $\mathbf{x}$ is in $V(\mathbf{y})$ implies $t\mathbf{x}$ is in $V(t\mathbf{y})$ for all $t > 0$.
3. $f(t\mathbf{x}) = tf(x)$ for all $t > 0$ i.e. $f(\mathbf{x})$is homogeneous of degree 1.

Increasing Return to Scale. A technology exhibits increasing return to scale if $f(t\mathbf{x}) > tf(x)$ for all $t > 1$.

Decreasing Return to Scale. A technology exhibits increasing return to scale if $f(t\mathbf{x}) < tf(x)$ for all $t > 1$.

Restricted Production Possibility sets may exhibits decreasing return to scale.

Consider what happens if we increase all inputs by some small amount t. This is given by:

$$\frac{\partial f(t\mathbf{x})}{\partial t}$$

We convert this into an elasticity:

$$\frac{\partial f(t\mathbf{x})}{\partial t}\frac{t}{f(t\mathbf{x})}$$

We evaluate this at $t = 1$ to see what elasticity of scale is at $t = 1$:

$$e(\mathbf{x}) = \frac{\partial f(t\mathbf{x})}{\partial t}\frac{t}{f(t\mathbf{x})}|_{t=1}$$

The elasticity of scale measures the percent increase in scale. We say that the technology exhibits increasing, constant or decreasing return to scale if $e(\mathbf{x})$ is greater, equal or less than 1.

1.5 The Competitive Firm

Consider the problem of a firm that takes prices as given in both its output and its factor markets. Let p be a vector of prices for inputs and outputs of the firm The profit maximization problem of the firm can be stated as:

$$\begin{aligned} \pi(\mathbf{p}) &= \max \mathbf{py} \\ & \quad s.t.\mathbf{y} \text{ is in } Y \end{aligned}$$

Since output are positive numbers and inputs are negative numbers, this problem give us the maximum of revenue minus costs. The $\pi(\mathbf{p})$ is the profit function of the firm.

In a short-run maximization problem we may define the short-run or restricted profit function:

$$\begin{aligned} \pi(\mathbf{p}, \mathbf{z}) &= \max \mathbf{py} \\ & \quad s.t.\mathbf{y} \text{ is in } Y(\mathbf{z}) \end{aligned}$$

If the firm produces only one output it may be written as:

$$\pi(p, \mathbf{w}) = \max pf(\mathbf{x}) - \mathbf{wx}$$

where p is now the scalar price of output and $\mathbf{w}$ is the vector of factor prices.

In this case we can also write a variant of the restricted profit function, the cost function:

$$\begin{aligned} c(\mathbf{w}, y) &= \min \mathbf{wx} \\ & \quad s.t.\mathbf{x} \text{ is in } V(y) \end{aligned}$$

In the short-run we want to consider the restricted or short-run cost function:

$$\begin{aligned} c(\mathbf{w},y,\mathbf{z}) &= \min \mathbf{wx} \\ & s.t.(y,-\mathbf{x})\text{is in } Y(z) \end{aligned}$$

The cost function gives the minimum cost of producing a level of output y when factor prices are $\mathbf{w}$.

Profit-maximizing and cost-minimizing behavior may be calculated.

The first-order conditions for the single output profit maximization problem are:

$$p\mathbf{D}f(\mathbf{x}^*) = \mathbf{w}$$

or

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = w_i$$

These conditions say that the value marginal product of each factor must be equal to its price.

The second-order condition for profit maximization is that the matrix of second order derivatives of the production function must be negative semidefinite at the optimum point, that is:

$$\mathbf{D}^2 f(\mathbf{x}^*) = \left(\frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j}\right)$$

satisfies the condition that $\mathbf{hD}^2 f(\mathbf{x}^*)\mathbf{h} \leq \mathbf{0}$ for all vectors $\mathbf{h}$. Geometrically this means that the production function must be locally concave in the neighborhood of an optimum.

For each vector of prices $(p, \mathbf{w})$ there will in general be some optimal choice of factors $\mathbf{x}^*$. The function $\mathbf{x}(p, \mathbf{w})$ that gives us the optimal choice of inputs as a function of the prices is called the demand function of the firm. Similarly, $y(p, \mathbf{w}) = f(\mathbf{x}(p, \mathbf{w}))$. is called the supply function of the firm.

We consider the problem of finding a cost-minimizing way to produce a given level of output:

$$\begin{aligned} &\min \mathbf{wx} \\ &s.t. f(\mathbf{x}) = y \end{aligned}$$

The Lagrangian expression for cost minimization is:

$$L(\mathbf{x},\lambda) = \mathbf{wx} - \lambda(f(\mathbf{x}) - y)$$

The first-order conditions characterizing an interior solution to this problem are:

$$\mathbf{w} = \boldsymbol{\lambda}\mathbf{D}f(\mathbf{x}^*)$$

where λ is the lagrange multiplier of the constraint
and the second order conditions :

$$\mathbf{h}\mathbf{D}^2 f(\mathbf{x}^*)\mathbf{h} \leq \mathbf{0} \text{ for all } \mathbf{h} \text{ satisfying } \mathbf{w}\mathbf{h} = 0$$

or:

$$\frac{\partial^2 L(\mathbf{x}^*, \lambda)}{\partial x_i \partial x_j} = -\lambda \left(\frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j} \right)$$

Algebraically, this means that $\mathbf{D}^2 f(\mathbf{x}^*)$ must be negative semidefinite for all directions $\mathbf{h}$ orthogonal to $\mathbf{w}$. The production function should be locally quasi-concave which is another way of saying that the input requirement sets must be locally convex.

From the first order conditions (since $\lambda = \frac{1}{w_i}\frac{\partial f(\mathbf{x}^*)}{\partial x_i}$for all i) we have that:

$$\frac{w_i}{w_j} = \frac{\frac{\partial f(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial f(\mathbf{x}^*)}{\partial x_j}} \quad i, j = 1, ...n$$

The term $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} \backslash \frac{\partial f(\mathbf{x}^*)}{\partial x_j}$ represents the technical rate of substitution at what rate factor j can be substituted for factor i while maintaining a constant level of output. The term $w_i \backslash w_j$ represents the economic rate of substitution at what rate factor j can be substituted for factor i.

For each choice of $\mathbf{w}$ and y there will be some choice of $\mathbf{x}^*$that minimizes the cost of producing y units of output. We will call the function that gives us this optimal choice the conditional factor demand function and write it as $\mathbf{x}(\mathbf{w}, y)$. That conditional demand depends on the level of output produced and on the factor prices.

We can combine the problem of cost minimization and profit maximization for a price-taking firm by writing the profit maximization problem in the following way:

$$\max py - c(\mathbf{w}, y)$$

Here the first term gives revenue and the second term the minimum cost of achieving that revenue. The first order conditions for profit maximization are:

$$p = \frac{\partial c(\mathbf{w}, y^*)}{\partial y}$$

or price equals marginal costs. The second order condition is $\partial^2 c(\mathbf{w}, y^*)/\partial y^2 \geqq 0$ or that marginal cost must be increasing at the optimal level of output.

1.6 Average and Marginal Costs

The cost function is our primary means of describing the economic possibilities of a firm.

The cost function can always be expressed as the value of the conditional factor demands:

$$c(\mathbf{w}, y) \equiv \mathbf{w}\mathbf{x}(\mathbf{w}, y)$$

In the short-run some of the factors of production are fixed at predetermined levels. Let $\mathbf{x}_f$ be the vector of fixed factors, and break up $\mathbf{w}$ into $\mathbf{w} = (\mathbf{w}_w, \mathbf{w}_f)$ the vector of prices of the variables and the fixed factors. The short-run conditional factor demand functions will generally depend on $\mathbf{x}_f$ so we write them as $\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)$.Then the short-run cost function may be written as:

$$c(\mathbf{w}, y, \mathbf{x}_f) \equiv \mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f\mathbf{x}_f$$

The term $\mathbf{w}_v\mathbf{x}(\mathbf{w}, y, \mathbf{x}_f)$ is called the short-run variable costs and the term $\mathbf{w}_f\mathbf{x}_f$ is fixed costs. We can define various derived cost concepts from these basic units:

$$SAC = \frac{c(\mathbf{w}, y, \mathbf{x}_f)}{y}$$

$$SAVC = \frac{\mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y}$$

$$SAFC = \frac{\mathbf{w}_f\mathbf{x}_f}{y}$$

$$STC = \mathbf{w}_v\mathbf{x}(\mathbf{w}, y, \mathbf{x}_f) + \mathbf{w}_f\mathbf{x}_f = c(\mathbf{w}, y, \mathbf{x}_f)$$

$$SMC = \frac{\partial c(\mathbf{w}, y, \mathbf{x}_f)}{\partial y}$$

When all factors are variable the firm will optimize in the choice of $\mathbf{x}_f$. The long-run cost function only depends on the factor prices and the level of output.

Let $\mathbf{x}_f(\mathbf{w}, y)$ be the optimal choice of the fixed factors and let $\mathbf{x}_v(\mathbf{w}, y) = \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f(\mathbf{w}, y))$ be the optimal choice of the variable factors. Then the long-run cost function may be written as:

$$c(\mathbf{w}, y) = \mathbf{w}_v\mathbf{x}_v(\mathbf{w}, y) + \mathbf{w}_f\mathbf{x}_f(\mathbf{w}, y) = c(\mathbf{w}, y), \mathbf{x}_f(\mathbf{w}, y))$$

The long-run cost function can be used to define:

$$LAC = \frac{c(\mathbf{w}, y)}{y}$$

$$LMC = \frac{\partial c(\mathbf{w}, y)}{\partial y}$$

1.7 The Geometry of Costs

In the short-run the cost function has two components: fixed costs and variable costs. We can write:

$$\begin{aligned} SAC &= \frac{c(\mathbf{w}, y, \mathbf{x}_f)}{y} = \frac{\mathbf{w}_f \mathbf{x}_f}{y} + \frac{\mathbf{w}_v \mathbf{x}_v(\mathbf{w}, y, \mathbf{x}_f)}{y} \\ &= SAFC + SAVC \end{aligned}$$

1.8 Long Run and Short Run Cost Curves

Let us write the long-run cost function as $c(y) = c(y, \mathbf{z}(y))$. Here we have omitted the factor prices since they are assumed fixed, and we let $\mathbf{z}(y)$ be the cost-minimizing demand for the fixed factors. Let y^* be some level of output and let $\mathbf{z}^* = \mathbf{z}(y^*)$ be the associated demand for the fixed factors. The short-run cost $c(y, \mathbf{z}^*)$ must be at least as greater as the long-run cost $c(y, \mathbf{z}(y))$ for all levels of output, and the short run cost will equal the long-run costs at output $y^* : c(y^*, \mathbf{z}^*) = c(y^*, \mathbf{z}(y^*))$. Hence the long run and the short run cost curves must be tangent at y^*. The slope of the long-run cost curve at $y^*$is:

$$\frac{dc(y^*, \mathbf{z}(y^*))}{dy} = \frac{\partial c(y^*, \mathbf{z}^*)}{dy} + \sum_i \frac{\partial c(y^*, \mathbf{z}^*)}{dy_i} \frac{\partial z_i(y^*)}{\partial y}$$

But since $\mathbf{z}^*$ is the optimal choice of the fixed factors at the output level y^*, we must have:

$$\frac{\partial c(y^*, \mathbf{z}^*)}{dy_i} = 0 \quad \text{all } i$$

Thus the long run marginal costs at y^* equal short-run marginal costs at $(y^*, \mathbf{z}^*)$.

1.9 Cost and Profit Function with Variable Factor Prices

Properties of the Cost Function:

1. (nondecreasing in $\mathbf{w}$)If $\mathbf{w}^{'} > \mathbf{w}$ then $c(\mathbf{w}^{'}, y) \geqslant c(\mathbf{w}, y)$
2. (homogeneous of degree 1 in $\mathbf{w}$) $c(t\mathbf{w}, y) = tc(\mathbf{w}, y)$ for $t > 0$
3. (concave in $\mathbf{w}$) $c(t\mathbf{w} + (1-t)\mathbf{w}', y) \geqslant tc(\mathbf{w}, y) + (1-t)c(\mathbf{w}', y)$
4. continuous in $\mathbf{w}$) c is continuous as a function of$\mathbf{w}$ for $\mathbf{w} \gg \mathbf{0}t$.

Properties of the Profit Function:

1. (nondecreasing in $\mathbf{p}$,nondecreasing in $\mathbf{w}$)If $\mathbf{p}^{'} \geqslant \mathbf{p}$ and $\mathbf{w}' \leq \mathbf{w}$ then $\pi(\mathbf{p}'\mathbf{w}') \geqslant \pi(\mathbf{p}, \mathbf{w})$
2. (homogeneous of degree 1 in $(\mathbf{p}, \mathbf{w})$) $\pi(t\mathbf{p}, t\mathbf{w}) = t\pi(\mathbf{p}, \mathbf{w})$ for $t > 0$

3. (convex in $\mathbf{p}, \mathbf{w}$) Let $(\mathbf{p}'', \mathbf{w}'') = (t\mathbf{p} + (1-t)\mathbf{p}', t\mathbf{w} + (1-t)\mathbf{w}')$

   Then:

   $\pi(\mathbf{p}'', \mathbf{w}'') \leq t\pi(\mathbf{p}, \mathbf{w}) + (1-t)\pi(\mathbf{p}'\mathbf{w}')$ for $0 \leq t \leq 1$

4. (continuous in $(\mathbf{p}, \mathbf{w})$) at least when $\mathbf{p} > 0$ and $\mathbf{w} \gg \mathbf{0}$.

1.10 Properties of Demand and Supply Functions

These functions give the optimal choices of inputs and outputs as a function of the prices are known as demand and supply functions.

The factor demand function $x_i(\mathbf{p}, \mathbf{w})$must satisfy the restriction that:

$$x_i(t\mathbf{p}, t\mathbf{w}) = x_i(\mathbf{p}, \mathbf{w})$$

so they must be homogeneous of degree zero.

A function $f : \mathbf{R}^n_+ \to \mathbf{R}$ is homogeneous of degree k if $f(t\mathbf{x}) = t^k f(\mathbf{x})$ for all $t > 0$. In particular f is homogeneous of degree 0 if $f(t\mathbf{x}) = f(\mathbf{x})$ and it is homogeneous of degree 1 if $f(t\mathbf{x}) = tf(\mathbf{x})$.

Euler's law: If $\mathbf{f}$ is a differentiable function homogeneous of degree 1 then:

$$f(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i$$

If $f(x)$ is homogeneous of degree 1 then $\frac{\partial f(x)}{\partial x_i}$ is homogeneous of degree 0.

1.11 The Relationship between Demand Function and Profit Function

Hotelling's Lemma:

Let $y(\mathbf{p}, \mathbf{w})$ be the firm's supply function and let $x_i(\mathbf{p}, \mathbf{w})$ be the firm's demand function for factor i. Then:

$$y(\mathbf{p}, \mathbf{w}) = \frac{\partial \pi(\mathbf{p}, \mathbf{w})}{\partial \mathbf{p}}$$

$$x_i(\mathbf{p}, \mathbf{w}) = -\frac{\partial \pi(\mathbf{p}, \mathbf{w})}{\partial \mathbf{w}}$$

when the derivative exists and when $\mathbf{w} \gg 0$ $\mathbf{p} \gg \mathbf{0}$.

When a price of a factor changes infinitesimally, there will be two effects. First, there is a direct effect on profits of $d\pi = -dw_i x_i(p, \mathbf{w})$ because if the price of a factor changes by 1 euro and the firm is employing 100 units of that factor , the profits will go down of 100 euros. Second, there is an indirect effect in that the firm will find in its interests to change its production plan . But the impact on profits of any infinitesimal change in the production plan must be zero since we are already at the optimum production plan. Hence, the total impact of the indirect effect is zero, and we are left only with the direct effect.

Shephard's lemma:

Let $x_i(\mathbf{w},y)$ be the firm's conditional factor demand for input i. Then if c is differentiable at $(\mathbf{w},y)$, and $\mathbf{w} \gg 0$

$$x_i(\mathbf{w},y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_i}$$

If we are operating at a cost-minimizing point and the price w_i increases, there will be a direct effect, in that the expenditure on the first factor will increase. There will also be an indirect effect, in that we will want to change the factor mix. But since we are operating at a cost-minimizing point, any such infinitesimal change must yield zero additional profits.

We have shown that cost function have certain properties that follow from the structure of the cost minimization properties; we have shown above that the demand function are simply the derivatives of the cost function. Hence, the properties we have found concerning the cost function will translate into certain restriction on its derivatives, the factor demand functions.

1. The cost function is increasing in factor prices. Therefore, $\frac{\partial c(\mathbf{w},y)}{\partial w_i} = x_i(\mathbf{w},y) \geqslant 0$.

2. The cost function is homogeneous of degree 1 in $\mathbf{w}$. Therefore, the derivative of the cost function, the factor demands, are homogeneous of degree 0 in $\mathbf{w}$.

3. The cost function is concave in $\mathbf{w}$.

Reminder:

A function $f : R^n \to R$ is concave if $f(t\mathbf{x} + (1-t)\mathbf{y}) \geqq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$ for all $0 \leqq t \leqq 1$. If f is twice differentiable we have equivalent criteria:

The function f is concave if $f(\mathbf{y}) \leqq f(\mathbf{x}) + \mathbf{D}f(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n. The function f is concave if the matrix of second derivatives $\mathbf{D}^2 f(\mathbf{x})$ is negative semidefinite at all $\mathbf{x}$.

1.12 Duality

We have seen if we are given a specification of the technology, say, as a production function, and if this technology satisfies certain stated properties, we can obtain its cost function under certain conditions. Could the exercise be carried out in reverse? That is, if we knew the cost function, could we infer the production technology that might have generated this cost function?

The answer is "yes, provided the technology is convex and monotonic." given the cost function, we can obtain a special form of the input requirement set as follows:

$$V^*(y) = \{\mathbf{x} | wx \geq \mathbf{w}\mathbf{x}(\mathbf{w}, y) = c(\mathbf{w}, y) \text{ for all } \mathbf{w} \geq 0\}$$

where the asterisk indicates that it is a special kind of input requirement set. We can now try and relate this input-requirement set, constructed on cost considerations, to the usual technologically defined input-requirement set, $V(y)$.

The claim is, if $V(y)$ represents a convex and monotonic technology, $V^*(y) = V(y)$. Further, if the technology is non-convex or non-monotonic, $V^*(y)$ will be a convexified, monotonized version of $V^*(y)$. The overall implication is that the cost function summarises all the economically relevant aspect of the underlying technology.