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Calibrating Mathematical Programming Spatial Models[1]

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1. Introduction

In the area of trade, modelers have a wide variety of tools at their disposal: spatial and non spatial partial equilibrium models, computable general equilibrium models. There is no superiority between them but rather a better adequacy or efficiency to deal with the specific issue at hand. Pros and cons of the different classes of models are addressed, among the others, in Anania (2001), Bouët (2008), Francois and Reinert (1997), and van Tongeren, van Meijl and Surry (2001). Partial equilibrium models tend to better accommodate explicit representations of complex policy instruments, allow for a more detailed representation of markets and require less restrictive assumptions. Computable general equilibrium models can deal with interdependence among sectors and income and employment effects.

In this paper we deal with spatial partial equilibrium models, that is with partial equilibrium models which are "naturally" able to reproduce bilateral trade flows without having to resort to the Armington assumption (Armington, 1969). These models are particularly useful when the market, or the markets, considered are relatively small with respect to the countries' overall economy and relevant trade policies include discriminatory instruments, that is policies which discriminate by country of origin (destination) of imports (exports), such as preferential tariffs, country specific tariff rate quotas or embargos. In particular, the focus of this paper is on mathematical programming spatial partial equilibrium models.

Empirical models of international trade are subject to a common pitfall that is represented by the discrepancy between actual and optimal trade flows, that is, between realized commodity flows in a given year and the import-export patterns generated by the model solution for the same year. In fact, mathematical programming models tend to suffer from an "overspecialization" of the optimal solution with respect to observed trade flows. The main reason for this discrepancy often originates with the transaction costs per unit of commodity bilaterally traded between two countries; generally this piece of crucial information is measured with a degree of imprecision which is well above that of other

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parameters in the model. When this event occurs, a calibration of the trade model for the given base year allows for more effective policy simulations. Different approaches have been used in the past to calibrate mathematical programming trade models, mostly based on including in the model additional constraints limiting the space of feasible solutions. The original calibration procedure proposed in this paper follows the approach used in Positive Mathematical Programming (PMP) (Howitt, 1995a and 1995b).

The paper is structured in two parts. The first part discusses the proposed calibration procedure with reference to a variety of spatial transportation and trade models; the second part provides numerical examples that support the implementation of the calibration procedure of the proposed models.
Regarding the types of trade models to be analyzed in the present paper, we distinguish between models that involve either one or two commodities and those that jointly involve three or more commodities. This distinction has to do with the integrability conditions of systems of demand functions.

When dealing with either one or two commodities, Samuelson, first, and Takayama and Judge, after him, have shown that the preferred specification of a spatial trade model among *R* regions corresponds to the maximization of a quasi welfare function (*QWF*) subject to two sets of constraints regarding the demand and the supply of the various regions. The *QWF* objective is defined as the integral of the inverse demand function(s) minus the integral of the supply function(s) and the total transaction costs.

As theory does not require symmetry of the Jacobian matrix in Marshallian systems of three or more demand functions, these systems are not integrable into a total gross revenue function and no suitable objective function is available for analyzing them. In these cases, the specification of an Equilibrium Problem will replace the formulation of a dual pair of optimization problems.

2. Calibrating Mathematical Programming Spatial Trade Models

2.1 The Classical Transportation Model

We begin with a simple transportation model involving *J* importing and *I* exporting countries. We assume a single homogeneous commodity whose quantities consumed by the *j*-th destination, $\bar{x}_j^D$, and supplied by the *i*-th origin, $\bar{x}_i^S$, are known together with the realized trade flow, $\bar{x}_{ij}$, and the fixed accounting transaction cost per unit of commodity, tc_{ij}, transported between country pairs. In all statements, indexes range as $i = 1,...,I$ and $j = 1,...,J$.

This simple model can be stated as follows:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij} x_{ij} \tag{1}$$

Dual variables

subject to $$\bar{x}_j^D \le \sum_{i=1}^{I} x_{ij} \qquad p_j^D \qquad (2)$$

$$\sum_{j=1}^{J} x_{ij} \le \bar{x}_i^S \qquad p_i^S \qquad (3)$$

and $x_{ij} \ge 0$. The interpretation of the dual variables p_j^D and p_i^S corresponds, respectively, to commodity prices at destination and at origin.

In general, transaction costs are estimated imprecisely, often extending the same unit cost to routes for which a direct figure is not available. An initial goal of the proposed calibration procedure, therefore, is to obtain a correct marginal transaction cost by means of a dual parameter, say λ_{ij}, that is consistent with the structure of the transportation model and the knowledge of realized trade flows. Thus, the corresponding linear programming model minimizes the total transaction cost, *TTC*, subject to conventional demand and supply constraints together with calibration constraints as in the following primal specification:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij} x_{ij} \qquad (4)$$

Dual variables

subject to $$\bar{x}_j^D \le \sum_{i=1}^{I} x_{ij} \qquad p_j^D \qquad (5)$$

$$\sum_{j=1}^{J} x_{ij} \le \bar{x}_i^S \qquad p_i^S \qquad (6)$$

$$x_{ij} = \bar{x}_{ij} \qquad \lambda_{ij} \qquad (7)$$

and $x_{ij} \ge 0$. λ_{ij} expresses the difference between the accounting and the effective transaction cost per unit of bilaterally traded commodity. While dual variables p_j^D and p_i^S are nonnegative by virtue of the specified direction of the associated constraints, nothing can be said *a priori* about the sign of dual variables λ_{ij} associated with calibration constraints (7). In fact, differently from the traditional PMP approach (Howitt, 1995a and 1995b), in this paper the calibrating constraints are stated as a set of equations, rather than inequalities. This means that either a reduction or an increase of the accounting – and, often, poorly measured – transaction cost is admissible. The specification of the calibration constraints admits the common event of “self-selection” that occurs when the realized trade between a given pair of countries is null. The economic justification for this occurrence is attributed to the "fact" that the marginal cost of trade is strictly greater than the associated marginal revenue.

The dual specification of the transportation model (4)-(7) is stated as the maximization of the net value added, *NVA*, of the transportation industry subject to the economic equilibrium constraints according to which its marginal cost per unit of commodity exchanged between a given pair of countries must be greater than or equal to its marginal revenue, that is

$$\max NVA = \sum_{j=1}^{J} p_j^D \overline{x}_j^D - \sum_{i=1}^{I} p_i^S \overline{x}_i^S - \sum_{i=1}^{I}\sum_{j=1}^{J} \lambda_{ij}\overline{x}_{ij} \qquad (8)$$

Dual variables

subject to $\quad p_j^D \le p_i^S + (tc_{ij} + \lambda_{ij}) \qquad x_{ij} \qquad (9)$

and $p_j^D \ge 0,\ p_i^S \ge 0,\ \lambda_{ij}$ free variable. The term $(tc_{ij} + \lambda_{ij})$ constitutes the effective transaction cost per unit of commodity transported from the ***i***-th to the ***j***-th countries. The supporting idea is that information about transaction costs is more difficult to obtain than information on trade flows. Hence, the utilization of all the available information – whether the accounting and, admittedly, imprecise transaction costs and the more accurate trade data – should provide a better specification of the international trade model. The level and the sign of the dual variable λ_{ij} resulted from the solution of model (4)-(7) will determine whether the accounting transaction cost tc_{ij} was originally either over- or under-estimated. The crucial realization, therefore, is that a solution of either the primal or the dual models defined above should not be regarded as a tautological statement but as a way to elicit the complete and more accurate marginal transaction costs to be used in subsequent analyses.

With knowledge of the dual variables λ_{ij} obtained from the solution of LP model (4)-(7), say λ_{ij}^*, a second phase LP model can be stated as follows:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J} (tc_{ij} + \lambda_{ij}^*) x_{ij} \qquad (10)$$

Dual variables

subject to $\quad \overline{x}_j^D \le \sum_{i=1}^{I} x_{ij} \qquad p_j^D \qquad (11)$

$$\sum_{j=1}^{J} x_{ij} \le \overline{x}_i^S \qquad p_i^S \qquad (12)$$

with $x_{ij} \ge 0$, $i = 1,...,I$ and $j = 1,...,J$.

Classical PMP modifies a linear objective function by adding a quadratic function which accounts for additional costs inferred based on the difference between the observed realization and the solution from the uncalibrated model. The calibration procedure proposed in this paper does not alter the objective function, but only "corrects" one set of its parameters (bilateral transaction costs). The classical PMP approach assumes that costs in the uncalibrated model can be only underestimated, while the approach proposed assumes that transaction costs can be either underestimated or overestimated (λ_{ij}^{*} are unrestricted). Classical PMP and the calibration procedure proposed here both assume the model is well specified in all its parts but in the parameters being subject to the calibration; this means, for example, that if the model is ill-designed with respect to the representation of existing policies, these errors will be captured by the λ_{ij}^{*} and subsequent policy simulations will yield distorted results.

The empirical solution of model (10)-(12) should be carried out using all the available information that includes the realized levels of activities. When the initial values of the trade flow variables are set equal to the realized level of trade flows, model (10)-(12) calibrates perfectly all its components. If initial values are set at levels different from the realized ones there is the possibility that the empirical model will detect alternative optimal trade flow matrices (Dantzig, 1951; Koopmans, 1947; Paris, 1981). However, the optimal solution always reproduces quantities consumed and produced in each country as well as demand and supply prices; this occurs because the structure of the objective functions at the optimum and that of the constraints is identical. To illustrate this assertion, let us specify the dual of model (10)-(12):

$$\max NVA = \sum_{j=1}^{J} p_j^D \bar{x}_j^D - \sum_{i=1}^{I} p_i^S \bar{x}_i^S \tag{13}$$

		Dual variables	
subject to	$p_j^D \le p_i^S + (tc_{ij} + \lambda_{ij}^{*})$	x_{ij}	(14)

with $p_j^D \ge 0,\ p_i^S \ge 0$. Constraints (5), (6) and (9) in the model with calibrating constraints are identical to constraints (11), (12) and (14) in the model without calibrating constraints. Furthermore, at the optimal solution the primal and dual objective functions in the two models are equal. This establishes the equivalence of the two specifications.

A more informative discussion about the correct adjustment appearing in the objective function of the calibrating model (10)-(12) involves the Lagrangean function of model (4)-(7):

$$L = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij}x_{ij} + \sum_{j=1}^{J} p_j^D(\bar{x}_j^D - \sum_{i=1}^{I} x_{ij}) + \sum_{i=1}^{I} p_i^S(\sum_{j=1}^{J} x_{ij} - \bar{x}_i^S) + \sum_{i=1}^{I}\sum_{j=1}^{J} \lambda_{ij}(x_{ij} - \bar{x}_{ij}) \tag{15}$$

with derivatives

$$\frac{\partial L}{\partial x_{ij}} = tc_{ij} - p_j^D + p_i^S + \lambda_{ij} \geq 0\,,\; i = 1,...,I;\, j = 1,...,J\;, \tag{16}$$

which indicate the correct adjustment of the per-unit transaction costs in the form of $p_j^D \leq (tc_{ij} + \lambda_{ij}) + p_i^S$, as given in constraints (9) and (14). Hence – because $\bar{x}_j^D$ and $\bar{x}_i^S$ are exogenously determined and the trade flows x_{ij} are the only variables – the objective function (10) expresses the desired and required parameterization for obtaining a set of multiple optimal solutions which contains the one that mimics the realized trade pattern.

The stylized nature of the above LP structures may be enriched with a more appropriate specification of an international trade model involving the paraphernalia of tariffs, subsidies, quotas, penalties, preferential trade treatments, exchange rates, etc. Hence, within reasonable parameter intervals, models (10)-(12) and (13)-(14) – augmented of the appropriate constraints – can be used to evaluate the likely effects of changes in policy interventions regarding tariffs, subsidies and other control parameters of interest.

2.2 A Model of International Trade with One Commodity

We assume R importing and exporting countries. When supply demand functions for each country are available, the classification between importing and exporting countries cannot be done in advance of solving the problem. Let us, therefore, define indices that cover all the regions (countries) without distinction between importers and exporters, $i, j = 1,...,R$. The known inverse demand function of the single commodity for the *j*-th country is assumed as $p_j^D = a_j - D_j x_j^D$, while the known inverse supply function for the same homogeneous commodity is assumed as $p_i^S = b_i + S_i x_i^S$. The coefficients a_j, D_j, S_i are known positive scalars. Parameter b_i is also known but may be either positive or negative. In this specification, quantities x_j^D and x_i^S are unknown and must be determined as part of the solution together with the trade flows x_{ij}. We assume, however, the availability of information concerning realized trade flows, $\bar{x}_{ij}$, and – as a consequence – knowledge of total quantities demanded, $\bar{x}_j^D$, and supplied, $\bar{x}_i^S$, for each country. We also assume knowledge (albeit imperfect) of the unit transaction costs, t_{ij}, $i, j = 1,...,R$.

The Samuelson-Takayama-Judge (STJ) model (Samuelson, 1952; Takayama and Judge, 1971) exhibits an objective function that maximizes a *QWF* function given by the difference between the areas below the inverse demand and above the inverse supply functions, diminished by total transaction costs. This specification corresponds to the maximization of the sum of consumer and producer surpluses netted out of total transaction costs.

The two elements of the *QWF* function – demand and supply functions, on one side, and total transaction costs, on the other side – may be subject to imprecise measurements. We assume here that only transaction costs are measured with imprecision. In fact, this is the crucial source for the discrepancy between realized and optimal traded quantities and total quantities demanded and supplied in each country, obtained from the solution of the STJ model.[2]

When information about the realized trade pattern, $\bar{x}_{ij}$, is available, a phase I PMP specification of the primal model takes on the following structure:

$$\max QWF = \sum_{j=1}^{R}(a_j - D_j x_j^D / 2)x_j^D - \sum_{i=1}^{R}(b_i + S_i x_i^S / 2)x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} tc_{ij}x_{ij} \qquad (17)$$

		Dual variables	
subject to	$x_j^D \leq \sum_{i=1}^{R} x_{ij}$	p_j^D	(18)
	$\sum_{j=1}^{R} x_{ij} \leq x_i^S$	p_i^S	(19)
	$x_{ij} = \bar{x}_{ij}$	λ_{ij}	(20)

and nonnegative variables, $x_j^D \geq 0, x_i^S \geq 0, x_{ij} \geq 0$, $(i, j = 1,2,\ldots,R)$.

The dual of model (17)-(20) may be stated as follows

$$\min TCMO = \sum_{j=1}^{R} x_j^D D_j x_j^D / 2 + \sum_{i=1}^{R} x_i^S S_j x_i^S / 2 + \sum_{i=1}^{R}\sum_{j=1}^{R} \lambda_{ij}\bar{x}_{ij} \qquad (21)$$

		Dual Variables	
subject to	$p_j^D \geq a_j - D_j x_j^D$	x_j^D	(22)
	$b_i + S_i x_i^S \geq p_i^S$	x_i^S	(23)
	$p_i^S + (tc_{ij} + \lambda_{ij}) \geq p_j^D$	x_{ij}	(24)

[2] Jansson and Heckelei (2009) propose a calibration procedure for mathematical programming spatial equilibrium models based on the estimation of transportation costs and prices, assumed to be stochastic, with measurement errors independent and identically distributed with known variances.

and $x_j^D \geq 0, x_i^S \geq 0, x_{ij} \geq 0$; λ_{ij} a free variable, $(i, j = 1, 2, \ldots, R)$. The economic interpretation of the objective function is given by the minimization of the total cost of market options and of the differential total transaction costs. When interpreting a dual model it is convenient to suppose that a second economic agent – external to the primal problem – desires to "take over the enterprise" of the primal economic agent. In this case, the dual agent will have to quote prices and quantities that will reimburse the primal agent of its "potential profit" (consumer and producer surpluses) plus the differential total transaction costs. The dual constraints express the demand and supply functions as well as the condition that the supply price in the *i*-th country plus the marginal effective transaction cost of the traded commodity between each pair of countries must be greater-than-or-equal to the demand price in the *j*-th country.

The solution of model (17)-(20) provides an estimate of the dual variables λ_{ij}^* associated with the calibration constraints that can be utilized in phase II of the PMP procedure for adjusting the unit transaction costs, as in the following calibrating model:[3]

$$\max QWF = \sum_{j=1}^{R} (a_j - D_j x_j^D / 2) x_j^D - \sum_{i=1}^{R} (b_i + S_i x_i^S / 2) x_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} (tc_{ij} + \lambda_{ij}^*) x_{ij} \qquad (25)$$

		Dual variables	
subject to	$x_j^D \leq \sum_{i=1}^{R} x_{ij}$	p_j^D	(26)
	$\sum_{j=1}^{R} x_{ij} \leq x_i^S$	p_i^S	(27)

and, $x_j^D \geq 0, x_i^S \geq 0, x_{ij} \geq 0$; $(i, j = 1, 2, \ldots, R)$. The adjustment of the unit transaction costs in (25) follows the same justification as presented in the previous section.

The Lagrangean function of problem (17)-(20) is:

[3] Bauer and Kasnakoglu (1990) used the PMP approach to calibrate a quadratic programming model of Turkish agriculture with endogeneous supply functions. Bouamra-Mechemache et al. (2002) calibrated a model similar to the one considered here by applying the classical PMP procedure (i.e. using inequality constraints to obtain the λ_{ij}^* and adding a quadratic cost function to the objective function); however, they found the calibrated solution not satisfactory and introduced further adjustments in the model.

$$L=\sum_{j=1}^{R}(a_j-D_jx_j^D/2)x_j^D-\sum_{i=1}^{R}(b_i+S_ix_i^S/2)x_i^S-\sum_{i=1}^{R}\sum_{j=1}^{R}tc_{ij}x_{ij}$$
$$+\sum_{j=1}^{R}p_j^D(\sum_{i=1}^{R}x_{ij}-x_j^D)+\sum_{i=1}^{R}p_i^S(x_i^S-\sum_{j=1}^{R}x_{ij})+\sum_{i=1}^{R}\sum_{j=1}^{R}\lambda_{ij}(\overline{x}_{ij}-x_{ij}) \qquad (28)$$

with relevant conditions:

$$\frac{\partial L}{\partial x_{ij}}=p_j^D-p_i^S-tc_{ij}-\lambda_{ij}\le 0\text{, and }\frac{\partial L}{\partial x_{ij}}x_{ij}=0\;. \qquad (29)$$

Model (25)-(27) calibrates total the observed total quantities demanded, $\overline{x}_j^D$, and supplied $\overline{x}_i^S$, in each country. When all the available information is fully exploited and the observed trade flows $\overline{x}_{ij}$ are used as initial values of the trade flow variables, the model calibrates perfectly. However, in general, a trade model may show multiple optimal solutions of trade flows, that is, solutions where different sets of trade flows are associated to the same quantities supplied and demanded in each country, the same total incurred adjusted transaction costs (calculated over all trade flows), and, as a result, the same value of the objective function. The possibility of multiple optimal solutions in terms of trade flows being associated to the unique optimal solution in terms of countries' net imports and exports does not come as a surprise because this is a common feature of this class of models (Dantzig, 1951; Koopsmans, 1947; Paris, 1983). In order to calibrate the observed trade flows, one needs just to use all the available information as initial starting values to guide the solver in search of the solution.

Let us assume now that only information about total demand, $\overline{x}_{r'}^D$, and total supply, $\overline{x}_r^S$, is available. The STJ model assumes the following specification:

$$\max QSW=\sum_{j=1}^{R}(a_j-D_jx_j^D/2)x_j^D-\sum_{i=1}^{R}(b_i+S_ix_i^S/2)x_i^S-\sum_{i=1}^{R}\sum_{j=1}^{R}tc_{ij}x_{ij} \qquad (30)$$

		Dual Variables	
subject to	$x_j^D\le\sum_{i=1}^{R}x_{ij}$	p_j^D	(31)
	$\sum_{j=1}^{R}x_{ij}\le x_i^S$	p_i^S	(32)
	$\overline{x}_j^D=x_j^D$	λ_j^D	(33)
	$x_i^S=\overline{x}_i^S$	λ_i^S	(34)

and, $x_j^D\ge 0, x_i^S\ge 0, x_{ij}\ge 0$, $(i, j=1,2,\dots,R)$.

The solution of model (30)-(34) provides an estimate of the dual variables associated with the calibration constraints, λ_j^{D*} and λ_i^{S*}, that can be utilized for adjusting the unit transaction costs as in the following calibrated model:

$$\max QWF = \sum_{j=1}^{R}(a_j - D_j x_j^D / 2)x_j^D - \sum_{i=1}^{R}(b_i + S_i x_i^S / 2)x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R}(tc_{ij} + \lambda_i^{S*} + \lambda_j^{D*})x_{ij} \qquad (35)$$

		Dual Variables	
subject to	$x_j^D \le \sum_{i}^{R} x_{ij}$	p_j^D	(36)
	$\sum_{j=1}^{R} x_{ij} \le x_i^S$	p_i^S	(37)

and, $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0$, $(i, j = 1, 2, \ldots, R)$.

The solution of model (35)-(37) calibrates exactly total demanded and supplied quantities in each country.

In order to justify the adjustments of the transaction costs in equations (35), the Lagrangean function of model (30)-(34) comes to the rescue:

$$\begin{aligned} L = &\sum_{j=1}^{R}(a_j - D_j x_j^D / 2)x_j^D - \sum_{i=1}^{R}(b_i + S_i x_i^S / 2)x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} tc_{ij}x_{ij} \\ &+ \sum_{j=1}^{R} p_j^D(\sum_{i=1}^{R} x_{ij} - x_j^D) + \sum_{i=1}^{R} p_i^S(x_i^S - \sum_{j=1}^{R} x_{ij}) \\ &+ \sum_{i=1}^{R}\lambda_i^S(\overline{x}_i^S - x_i^S) + \sum_{j=1}^{R}\lambda_j^D(\overline{x}_j^D - x_j^D) \end{aligned} \qquad (38)$$

with relevant conditions

$$\frac{\partial L}{\partial x_{ij}} = -tc_{ij} + p_j^D - p_i^S - \lambda_i^S - \lambda_j^D \le 0, \quad \text{or} \quad p_j^D \le p_i^S + tc_{ij} + \lambda_i^S + \lambda_j^D \ , \qquad (39)$$

and $$\frac{\partial L}{\partial x_{ij}} x_{ij} = 0 \qquad (40)$$

These conditions define the adjusted per-unit transaction cost ($tc_{ij} + \lambda_i^S + \lambda_j^D$) needed for the model to calibrate observed demanded and supplied quantities.[4]

2.3 A Multi-Commodity Samuelson-Takayama-Judge Model of International Trade

The extension of international trade models to multi-commodity exchanges requires a substantial adjustment to the structure of the mathematical programming models discussed above. First of all, it requires a considerably larger quantity of information that, if and when available, imposes the need of a careful management. The major shift from previous models is constituted by the specification of systems of demand and supply functions for each country. It follows that a properly defined system of demand and supply functions – for each country involved in the commodity exchange – ought to exhibit full matrices of demand and supply cross-price parameters. This is a formidable information requirement that, when overcome, may produce adequate empirical results as well as sensible policy analyses. Secondly, a special comment regards matrices $\mathbf{D}_j$ and $\mathbf{S}_j$, the matrices of cross-derivatives of the *j*-th country systems of demand and supply functions. In principle, demand and production theory requires neither the symmetry nor the positive semidefinitess of such matrices. However, the specification of a STJ problem in the form of maximizing a *QWF* objective function that assumes a quadratic structure would impose the requirement that the matrices $\mathbf{D}_j$ and $\mathbf{S}_j$ be symmetric and positive semidefinite. This is quite a strong assumption, since there is no reason why $\mathbf{D}_j$ and $\mathbf{S}_j$ should satisfy these conditions. Hence in section 2.4 we will present and discuss a structure, called the Equilibrium Problem, that will admit asymmetric $\mathbf{D}_j$ and $\mathbf{S}_j$ matrices.

We assume *K* homogeneous commodities, $K \geq 3$. Each country owns a system of *K* inverse demand functions, $\mathbf{p}_j^D = \mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D$, j = 1,…, R , and an inverse system of *K* inverse supply functions, $\mathbf{p}_j^S = \mathbf{b}_j + \mathbf{S}_j \mathbf{x}_j^S, \quad j = 1,...,R$. The matrix of nominal unit transaction costs is defined in three dimensions as $\boldsymbol{T} = \left[tc_{ijk} \right],\ i, j = 1,...,R,\ k = 1,...,K$ where $\mathbf{tc}_{ij}$ is the vector of unit transaction costs from country *i* to country *j* for the *K* commodities and $\mathbf{tc}_{jj}$ is the vector of domestic transaction costs in country *j*. We assume that information about the trade pattern for all commodities, $\overline{\mathbf{x}}_{ij}$, and, hence, total demands, $\overline{\mathbf{x}}_j^D$, and total supplies, $\overline{\mathbf{x}}_i^D$, is available for a given base year.

2.3.1 Case 1: demand and supply functions are well measured at different market levels

[4] Based on (39), an alternative interpretation of the role played by the λ_i^S and λ_j^D parameters could be in terms of adjustments of the intercepts of supply and demand functions.

We will consider two different cases. First inverse demand and supply functions are measured at different levels, e.g. the supply function at the farm gate and the demand function at retail, and the only information in the model which is measured with imprecision are transaction costs.

Except for the dimensionality of the price, quantity and transaction cost vectors, the corresponding STJ model exhibits a structure that is similar to that of model (17)-(20):

$$\max QWF = \sum_{j=1}^{R} (\mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D / 2)' \mathbf{x}_j^D - \sum_{i=1}^{R} (\mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S / 2)' \mathbf{x}_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} \mathbf{tc}'_{ij} \mathbf{x}_{ij} \quad (41)$$

Dual variables

subject to

$$\mathbf{x}_j^D \le \sum_{i=1}^{R} \mathbf{x}_{ij} \qquad \mathbf{p}_j^D \quad (42)$$

$$\sum_{j=1}^{R} \mathbf{x}_{ij} \le \mathbf{x}_i^S \qquad \mathbf{p}_i^S \quad (43)$$

$$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij} \qquad \boldsymbol{\lambda}_{ij} \quad (44)$$

All variables are nonnegative. The dual of model (41)-(44) is obtained in the usual fashion, by formulating the associated Lagrangean function, deriving the Karush-Kuhn-Tucker (KKT) conditions and, furthermore, by simplifying the Lagrangean function, which assumes the role of objective function in the dual problem.

$$\min TCMO = \sum_{j=1}^{R} \mathbf{x}'^D_j \mathbf{D}_j \mathbf{x}_j^D / 2 + \sum_{i=1}^{R} \mathbf{x}'^S_i \mathbf{S}_i \mathbf{x}_i^S / 2 + \sum_{i=1}^{R} \sum_{j=1}^{R} \overline{\mathbf{x}}'_{ij} \boldsymbol{\lambda}_{ij} \quad (45)$$

Dual Variables

subject to

$$\mathbf{p}_j^D \ge \mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D \qquad \mathbf{x}_j^D \quad (46)$$

$$\mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S \ge \mathbf{p}_i^S \qquad \mathbf{x}_i^S \quad (47)$$

$$\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) \ge \mathbf{p}_j^D \qquad \mathbf{x}_{ij} \quad (48)$$

All variables are nonnegative except $\boldsymbol{\lambda}_{ij}$ which is regarded as a vector of *K* free variables. The economic interpretation of model (45)-(48) is similar to that one given for dual model (21)-(24).

The solution of model (41)-(44) provides estimates of dual variables $\boldsymbol{\lambda}_{ij}$, say $\boldsymbol{\lambda}^*_{ij}$, that can be used to define effective transaction costs along the line of the PMP methodology proposed above. Hence, the calibrating STJ model for this more general international trade specification can be assembled as in the following structure

$$\max QSW = \sum_{j=1}^{R}(\mathbf{a}_j - \mathbf{D}_j\mathbf{x}_j^D / 2)'\mathbf{x}_j^D - \sum_{i=1}^{R}(\mathbf{b}_i + \mathbf{S}_i\mathbf{x}_i^S / 2)'\mathbf{x}_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^*)'\mathbf{x}_{ij} \tag{49}$$

Dual variables

subject to $\sum_{i=1}^{R}\mathbf{x}_{ij} \ge \mathbf{x}_j^D$ $\quad\mathbf{p}_j^D$ (50)

$\sum_{j=1}^{R}\mathbf{x}_{ij} \le \mathbf{x}_i^S$ $\quad\mathbf{p}_i^S$ (51)

with all variables nonnegative. The solution of model (49)-(51) will calibrate precisely the realized demanded and supplied quantities.

Extension 1: Estimation of Systems of Demand and Supply Functions

When information about the vectors of total demand quantities, $\overline{\mathbf{x}}_{jt}^D$, and supply quantities, $\overline{\mathbf{x}}_{it}^S$, is available for a number of T years – together with the corresponding demand prices, $\overline{\mathbf{p}}_{jt}^D$, and supply prices, $\overline{\mathbf{p}}_{it}^S$, $t = 1,...,T$, it is possible to estimate systems of demand and supply functions for each country. This estimation is performed in the same spirit of PMP; it attempts to utilize – and exploit in a logical and consistent way – all the available information.

Demand Functions

A least-squares approach is proposed for estimating the system of demand functions. In order to satisfy the integrability condition – which admits the definition of the proper objective function for the STJ model – the estimation is subject to the symmetry of the matrix of cross-derivatives, $\mathbf{D}_j$, as well as to its positive semidefiniteness. Hence,

$$\min \sum_{t=1}^{T}(\mathbf{u}_{jt}^D)'\mathbf{u}_{jt}^D \tag{52}$$

subject to

$$\overline{\mathbf{p}}_{jt}^D = \mathbf{a}_j - \mathbf{D}_j\overline{\mathbf{x}}_{jt}^D + \mathbf{u}_{jt}^D \tag{53}$$

$$\mathbf{D}_j = \mathbf{L}_j\boldsymbol{\Theta}_j\mathbf{L}_j' \tag{54}$$

$$\sum_{t=1}^{T}\mathbf{u}_{jt}^D = \mathbf{0} \tag{55}$$

with $\Theta_{j,k,k} \geq 0$. Constraint (53) specifies the system of demand functions. Constraint (54) defines the Cholesky factorization that generates the symmetry and the positive semidefiniteness of the $\mathbf{D}_j$ matrix. The matrix $\mathbf{L}_j$ is a unit lower triangular matrix while the matrix $\mathbf{\Theta}_j$ is a diagonal matrix with all nonnegative elements that guarantee the positive semidefiniteness of the $\mathbf{D}_j$ matrix. Constraint (55) guarantees that all the year deviations add up to zero.

The interpretation of the term $\mathbf{u}_{jt}^{D}$ deserves a special comment. Within the context of a calibrating PMP approach, and under the assumption that only information for a very limited number of years is available, it is convenient to interpret this term as a yearly deviation from the average system of demand functions rather than as either an "error" or a "disturbance term." In other words, the yearly realization of the demand prices in the *r*-th region would deviate from the average prices by the amount $\mathbf{u}_{rt}^{D}$. Knowledge of this deviation, therefore, is crucial for assuring the calibration of the model over every region and every year.

An analogous approach may be used to estimate the system of supply functions.

Extension 2: A Multi-Year STJ Model of International Trade

With the estimation of the demand and supply systems, a PMP model may be specified over *T* years along the lines presented in equations (52)-(55). Thus, assuming that information about the trade flows in each year, $\overline{\mathbf{x}}_{ijt}$, is available:

$$\begin{aligned}\max QWF = &\sum_{t=1}^{T}\sum_{j=1}^{R}(\hat{\mathbf{a}}_j - \hat{\mathbf{D}}_j\mathbf{x}_{jt}^{D}/2 + \hat{\mathbf{u}}_{jt}^{D})'\mathbf{x}_{jt}^{D} \\ &- \sum_{t=1}^{T}\sum_{i=1}^{R}(\hat{\mathbf{b}}_i + \hat{\mathbf{S}}_i\mathbf{x}_{it}^{S}/2 + \hat{\mathbf{u}}_{it}^{S})'\mathbf{x}_{it}^{S} - \sum_{t=1}^{T}\sum_{i=1}^{R}\sum_{j=1}^{R}\mathbf{tc}'_{ijt}\mathbf{x}_{ijt}\end{aligned} \tag{56}$$

		Dual variables	
subject to	$\sum_{i=1}^{R}\mathbf{x}_{ijt} \geq \mathbf{x}_{jt}^{D}$	$\mathbf{p}_{jt}^{D}$	(57)
	$\sum_{j=1}^{R}\mathbf{x}_{ijt} \leq \mathbf{x}_{it}^{S}$	$\mathbf{p}_{it}^{S}$	(58)
	$\mathbf{x}_{ijt} = \overline{\mathbf{x}}_{ijt}$	λ_{ijt}	(59)

with all nonnegative variables, but λ_{ijt} which is unrestricted. This first phase model provides the essential estimates of the dual variables λ_{ijt}, say λ^*_{ijt}. Therefore, the calibrating PMP model can be specified as follows

$$\max QWF = \sum_{t=1}^{T}\sum_{j=1}^{R}(\hat{\mathbf{a}}_j - \hat{\mathbf{D}}_j\mathbf{x}_{jt}^D/2 + \hat{\mathbf{u}}_{jt}^D)'\mathbf{x}_{jt}^D$$
$$-\sum_{t=1}^{T}\sum_{i=1}^{R}(\hat{\mathbf{b}}_i + \hat{\mathbf{S}}_i\mathbf{x}_{it}^S/2 + \hat{\mathbf{u}}_{it}^S)'\mathbf{x}_{it}^S - \sum_{t=1}^{T}\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ijt} + \lambda_{ijt}^*)'\mathbf{x}_{ijt} \qquad (60)$$

		Dual variables	
subject to	$\sum_{i=1}^{R}\mathbf{x}_{ijt} \geq \mathbf{x}_{jt}^D$	$\mathbf{p}_{jt}^D$	(61)
	$\sum_{j=1}^{R}\mathbf{x}_{ijt} \leq \mathbf{x}_{it}^S$	$\mathbf{p}_{it}^S$	(62)

with all nonnegative variables. The above model calibrates the quantities demanded and supplied in each country.

2.3.2 *Case 2: demand and supply functions are measured with imprecision at the same market level*

We consider a second case where demand and supply functions are measured at the same market level - the retail one - and transaction costs as well as demand and supply functions are measured with imprecision. This means that $\mathbf{tc}_{jj}$, the vector of domestic transaction costs in country *j*, is the null vector and $\mathbf{p}_j^D = \mathbf{p}_j^S$, for all *j* =1, 2, …, *R* .

This is the case where the calibration procedure, makes the model reproduce observed trade patterns, and, at the same time, adjusts parameters of the demand and supply functions to make these and observed trade flows, $\bar{\mathbf{x}}_{ij}$, consistent with the condition $\mathbf{p}_j^D = \mathbf{p}_j^S$.

In this case in phase I, a least-squares approach is proposed to estimate simultaneously the adjustments of transaction costs and of demand and supply function parameters needed for the model to reproduce observed trade patterns and comply with the condition that supply prices equal demand prices. In order to satisfy the conditions needed for the definition of the proper objective function for the STJ model, the estimation takes into account the need to assure the symmetry and positive semidefiniteness of the adjusted matrices of cross-derivatives.

The objective function is composed of the Sum of Squared Residuals of the intercepts and slopes of the demand and supply functions, plus a primal-dual component that represents the combination of the dual objective function of the problem minus the primal objective function. At the optimum this primal-dual portion of the objective function should achieve the value of zero. The constraints combine primal and dual constraints.

Using the familiar notation the model can be specified as follows

$$\begin{aligned}\min LS = &\sum_{j=1}^{R}\mathbf{u}_j{}'\mathbf{u}_j/2+\sum_{i=1}^{R}\mathbf{v}_i{}'\mathbf{v}_i/2+\sum_{j=1}^{R}trace(\mathbf{W}'_j\mathbf{W}_j)/2+\sum_{i=1}^{R}trace(\mathbf{Y}'_i\mathbf{Y}_i)/2+\\ &\{\sum_{j=1}^{R}\sum_{i=1}^{R}\bar{\mathbf{x}}_{ij}\boldsymbol{\lambda}_{ij}-[\sum_{j=1}^{R}((\mathbf{a}_j+\mathbf{u}_j)-(\mathbf{D}_j+\mathbf{W}_j)\ \mathbf{x}_j^D)'\mathbf{x}_j^D-\sum_{i=1}^{R}((\mathbf{b}_i+\mathbf{v}_i)+(\mathbf{S}_i+\mathbf{Y}_i)\ \mathbf{x}_i^S)'\mathbf{x}_i^S \\ &-\sum_{i=1}^{R}\sum_{j=1}^{R}\mathbf{tc}'_{ij}\mathbf{x}_{ij}]\}\end{aligned} \tag{63}$$

subject to

$$\sum_{i=1}^{R}\mathbf{x}_{ij}\geq\mathbf{x}_j^D \tag{64}$$

$$\sum_{j=1}^{R}\mathbf{x}_{ij}\leq\mathbf{x}_i^S \tag{65}$$

$$\mathbf{x}_{ij}=\bar{\mathbf{x}}_{ij} \tag{66}$$

$$\mathbf{W}_j=\mathbf{L}_j\boldsymbol{\Theta}_j\mathbf{L}'_j \tag{67}$$

$$\mathbf{Y}_i=\mathbf{M}_i\boldsymbol{\Phi}_i\mathbf{M}_i{}' \tag{68}$$

$$\mathbf{p}_j^D\geq(\mathbf{a}_j+\mathbf{u}_j)-(\mathbf{D}_j+\mathbf{W}_j)\mathbf{x}_j^D \tag{69}$$

$$(\mathbf{b}_i+\mathbf{v}_i)+(\mathbf{S}_i+\mathbf{Y}_i)\mathbf{x}_i^S\geq\mathbf{p}_i^S \tag{70}$$

$$\mathbf{p}_i^S+(\mathbf{tc}_{ij}+\boldsymbol{\lambda}_{ij})\geq\mathbf{p}_j^D \tag{71}$$

$$\mathbf{p}_j^S=\mathbf{p}_j^D \tag{72}$$

with $\Theta_{j,k,k}\geq 0$, $\Phi_{i,k,k}\geq 0$. The matrices $\mathbf{L}_j$ and $\mathbf{M}_i$ are unit lower triangular matrices, while matrices $\boldsymbol{\Theta}_j$ and $\boldsymbol{\Phi}_i$ are diagonal matrices with all nonnegative elements Constraints (67) and (68) define the Cholesky factorization that generates the symmetry and positive semidefiniteness of the $\mathbf{W}_j$ and $\mathbf{Y}_i$ matrices, a sufficient, although not necessary, condition for the symmetry and semidefiniteness of the matrices of the adjusted slopes, ($\mathbf{D}_j$ + $\mathbf{W}_j$) and ($\mathbf{S}_i$ + $\mathbf{Y}_i$), in the systems of demand and supply functions.

The phase II calibrating model takes on the familiar maximization structure:

$$\begin{aligned}\max QWF = &\sum_{j=1}^{R}((\mathbf{a}_j+\hat{\mathbf{u}}_j)-(\mathbf{D}_j+\hat{\mathbf{W}}_j)\mathbf{x}_j^D/2)'\mathbf{x}_j^D-\sum_{i=1}^{R}((\mathbf{b}_i+\hat{\mathbf{v}}_i)+(\mathbf{S}_i+\hat{\mathbf{Y}}_i)\mathbf{x}_i^S/2)'\mathbf{x}_i^S\\ &-\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij}+\hat{\boldsymbol{\lambda}}_{ij})'\mathbf{x}_{ij}\end{aligned} \tag{73}$$

subject to

$$\sum_{i=1}^{R} \mathbf{x}_{ij} \geq \mathbf{x}_j^D \tag{74}$$

$$\sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S \tag{75}$$

where $\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_i, \hat{\mathbf{W}}_j, \hat{\mathbf{Y}}_i$ and $\hat{\boldsymbol{\lambda}}_{ij}$ are the least-squares estimates obtained in phase I of the corresponding parameters.

This model calibrates demanded and supplied quantities and assures that the prices of demand are equal to the prices of supply in each region.

2.4 The Equilibrium Problem

When matrices $\mathbf{D}_j$ and the $\mathbf{S}_j$ are not symmetric, the system of demand and supply functions cannot be integrated and no suitable objective function exists for the STJ-type model. The Equilibrium Problem constitutes the appropriate mathematical programming structure for analyzing this trade scenario.

Definition
Let us consider the demand (*Dem*) and supply (*Sup*) of a commodity with quantity (*Q*), price (*P*) and marginal cost (*MC*). Then, the Equilibrium Problem is jointly defined by the following two sets of relations:

$$\text{Primal: } P \geq 0, \qquad Dem \leq Sup, \qquad (Sup - Dem)P = 0 \tag{76}$$

$$\text{Dual: } Q \geq 0 \qquad MC \geq P, \qquad (MC - P)Q = 0 \tag{77}$$

Hence, the phase I Equilibrium Problem with systems of demand and supply functions whose matrices $\mathbf{D}_j$ and $\mathbf{S}_i$ are not assumed to be symmetric is specified as follows:

Primal relations:

$$\mathbf{p}_j^D \geq \mathbf{0}, \qquad \mathbf{x}_j^D \leq \sum_{i=1}^{R} \mathbf{x}_{ij}, \qquad (\sum_{i=1}^{R} \mathbf{x}_{ij} - \mathbf{x}_j^D)'\mathbf{p}_j^D = \mathbf{0} \tag{78}$$

$$\mathbf{p}_i^S \geq \mathbf{0}, \qquad \sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S, \qquad (\mathbf{x}_i^S - \sum_{j=1}^{R} \mathbf{x}_{ij})'\mathbf{p}_i^S = \mathbf{0} \tag{79}$$

$$\boldsymbol{\lambda}_{ij} \text{ free}, \qquad \mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij}, \qquad (\overline{\mathbf{x}}_{ij} - \mathbf{x}_{ij})'\boldsymbol{\lambda}_{ij} = \mathbf{0} \tag{80}$$

Dual relations:

$$\mathbf{x}_j^D \geq \mathbf{0}, \qquad \mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D \leq \mathbf{p}_j^D, \qquad (\mathbf{p}_j^D - \mathbf{a}_j + \mathbf{D}_j \mathbf{x}_j^D)'\mathbf{x}_j^D = \mathbf{0} \tag{81}$$

$$\mathbf{x}_i^S \geq \mathbf{0}, \qquad \mathbf{p}_i^S \leq \mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S, \qquad (\mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S - \mathbf{p}_i^S)'\mathbf{x}_i^S = \mathbf{0} \tag{82}$$

$$\mathbf{x}_{ij} \geq \mathbf{0}, \quad \mathbf{p}_j^D \leq \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}), \quad [\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) - \mathbf{p}_j^D]'\mathbf{x}_{ij} = \mathbf{0}. \tag{83}$$

The asymmetry of the $\mathbf{D}_j$ and $\mathbf{S}_i$ matrices causes neither theoretical nor computational difficulties since the systems of demand and supply functions appear directly into the dual constraints (81) and (82) without the need of passing through an integral of the system – that does not exist in this case – and the corresponding (not existent) primal objective function.

2.4.1 Case 1: imprecision with transaction costs, demand and supply functions are measured at different market levels

As we did for the STJ model in section 2.3, we also consider two different cases for the Equilibrium Problem.

When parameter imprecision is assumed to regard only transaction costs, the solution of Equilibrium Problem (78)-(83) can be obtained by introducing primal and dual slack variables into the structural constraints and exploiting the complementary slackness conditions – that add up to zero – in the form of an auxiliary objective function to be minimized, since each term is nonnegative. Thus, using nonnegative slack variables $\mathbf{z}_{jP1}, \mathbf{z}_{iP2}, \mathbf{z}_{jD1}, \mathbf{z}_{iD2}, \mathbf{z}_{ijD3}$, (where the subscript of $\mathbf{z}_{jP1}, \mathbf{z}_{iP2}$ stands for primal constraints 1 and 2 and the subscript of $\mathbf{z}_{jD1}, \mathbf{z}_{iD2}, \mathbf{z}_{ijD3}$ stands for dual constraints 1, 2 and 3) the solution of the phase I Equilibrium Problem can be obtained by solving the following specification:

$$\min\{\textstyle\sum_{ij}[\mathbf{z}'_{jP1}\mathbf{p}_j^D + \mathbf{z}'_{iP2}\mathbf{p}_i^S + \mathbf{z}'_{jD1}\mathbf{x}_j^D + \mathbf{z}'_{iD2}\mathbf{x}_i^S + \mathbf{z}'_{ijD3}\mathbf{x}_{ij}]\} \tag{84}$$

		Dual variables	
subject to	$\mathbf{x}_j^D + \mathbf{z}_{jP1} = \sum_{i=1}^{R} \mathbf{x}_{ij}$,	$\mathbf{p}_j^D \geq \mathbf{0}$	(85)
	$\sum_{i=1}^{R} \mathbf{x}_{ij} + \mathbf{z}_{iP2} = \mathbf{x}_i^S$,	$\mathbf{p}_i^S \geq \mathbf{0}$	(86)
	$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij}$,	λ_{ij} free	(87)
	$\mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D + \mathbf{z}_{jD1} = \mathbf{p}_j^D$,	$\mathbf{x}_j^D \geq \mathbf{0}$	(88)
	$\mathbf{p}_i^S + \mathbf{z}_{iD2} = \mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S$,	$\mathbf{x}_i^S \geq \mathbf{0}$	(89)
	$\mathbf{p}_j^D + \mathbf{z}_{ijD3} = \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \lambda_{ij})$,	$\mathbf{x}_{ij} \geq \mathbf{0}$.	(90)

One advantage of this mathematical programming specification is that the optimal value of the objective function is known and it is equal to zero. Once again, the crucial task of a phase I Equilibrium Problem is to obtain consistent estimates of the dual variables λ_{ij}

associated to the calibrating constraint (87), say $\boldsymbol{\lambda}^*_{ij}$. With such estimates, a calibrating Equilibrium Problem can be stated as the following Phase II specification:

$$\min\{\sum_{ij}\mathbf{z}'_{jP1}\mathbf{p}_j^D+\mathbf{z}'_{iP2}\mathbf{p}_i^S+\mathbf{z}'_{jD1}\mathbf{x}_j^D+\mathbf{z}'_{iD2}\mathbf{x}_i^S+\mathbf{z}'_{ijD3}\mathbf{x}_{ij}\} \tag{91}$$

Dual variables

subject to,

$$\mathbf{x}_j^D+\mathbf{z}_{jP1}=\sum_{i=1}^{R}\mathbf{x}_{ij}\,, \qquad \mathbf{p}_j^D\geq\mathbf{0} \tag{92}$$

$$\sum_{j=1}^{R}\mathbf{x}_{ij}+\mathbf{z}_{iP2}=\mathbf{x}_i^S\,, \qquad \mathbf{p}_i^S\geq\mathbf{0} \tag{93}$$

$$\mathbf{a}_j-\mathbf{D}_j\mathbf{x}_j^D+\mathbf{z}_{jD1}=\mathbf{p}_j^D\,, \qquad \mathbf{x}_j^D\geq\mathbf{0} \tag{94}$$

$$\mathbf{p}_i^S+\mathbf{z}_{iD2}=\mathbf{b}_i+\mathbf{S}_i\mathbf{x}_i^S\,, \qquad \mathbf{x}_i^S\geq\mathbf{0} \tag{95}$$

$$\mathbf{p}_j^D+\mathbf{z}_{ijD3}=\mathbf{p}_i^S+(\mathbf{tc}_{ij}+\boldsymbol{\lambda}^*_{ij})\,, \qquad \mathbf{x}_{ij}\geq\mathbf{0}\,. \tag{96}$$

This calibrating model can now be used to estimate the response to changes in specific policy measures.

2.4.2 Case 2: imprecision of unit transaction costs and demand and supply functions (demand and supply are measured at the same market level)

When demand and supply functions are measured at the same market level and are inconsistent with the condition that, $\bar{\mathbf{x}}_{ij}=\mathbf{x}^*_{ij}$, and with that ($\mathbf{p}_j^D=\ \mathbf{p}_j^S$) for j =1, 2, …, R as required by theory, we assume that such demand and supply functions as well as unit transaction costs, are measured with imprecision. As a remedy, we associate vectors and matrices of deviations to both the intercepts and the slopes of the supply and demand functions, as well as to the unit transaction costs. All these deviations are jointly estimated in a least-squares model subject to appropriate constraints.

The relevant phase I model can be specified as follows: vectors $\mathbf{u}_j$ and $\mathbf{v}_i$ are unrestricted adjustments to the intercept vectors defining the demand and the supply functions, respectively. Similarly, matrices $\mathbf{W}_j$ and $\mathbf{Y}_i$ are unrestricted adjustments to the slope matrices defining demand and supply functions, respectively. All these parameters will be estimated by a least-squares approach subject to the economic relationships of the equilibrium problem. The complementary slackness conditions of the equilibrium problem (which are equal to zero) will appear in the objective function together with the sums of squared adjustments:

$$\min LS = \sum_{j=1}^{R} \mathbf{u}_j{}'\mathbf{u}_j / 2 + \sum_{i=1}^{R} \mathbf{v}_i{}'\mathbf{v}_i / 2 + \sum_{j=1}^{R} trace(\mathbf{W}'_j \mathbf{W}_j) / 2 + \sum_{i=1}^{R} trace(\mathbf{Y}'_i \mathbf{Y}_i) / 2 +$$
$$\{\sum_{j=1}^{R}\sum_{i=1}^{R} \bar{\mathbf{x}}_{ij}\boldsymbol{\lambda}_{ij} - [\sum_{j=1}^{R}((\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\ \mathbf{x}_j^D)'\mathbf{x}_j^D -$$
$$\sum_{i=1}^{R}((\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\ \mathbf{x}_i^S)'\mathbf{x}_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} \mathbf{tc}'_{ij}\mathbf{x}_{ij}]\} \tag{97}$$

subject to

$$\sum_{i=1}^{R} \mathbf{x}_{ij} \geq \mathbf{x}_j^D \tag{98}$$
$$\sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S \tag{99}$$
$$\mathbf{x}_{ij} = \bar{\mathbf{x}}_{ij} \tag{100}$$
$$\mathbf{p}_j^D \geq (\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\mathbf{x}_j^D \tag{101}$$
$$(\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\mathbf{x}_i^S \geq \mathbf{p}_i^S \tag{102}$$
$$\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) \geq \mathbf{p}_j^D \tag{103}$$
$$\mathbf{p}_j^S = \mathbf{p}_j^D \tag{104}$$

The complementary slackness conditions of the equilibrium problem appear in the portion of equation (97) within the curly brackets which should achieve a zero value when an optimal solution is reached. The remaining components of the objective function are the sums of squared deviations.

The phase II calibrated model includes the estimates of the adjustment coefficients obtained in phase I ($\hat{\mathbf{u}}_j$, $\hat{\mathbf{v}}_i$, $\hat{\mathbf{W}}_j$, $\hat{\mathbf{Y}}_i$, and $\hat{\boldsymbol{\lambda}}_{ij}$), in the minimization structure of the Equilibrium Problem:

$$\min\{\textstyle\sum_{ij} \mathbf{z}'_{jP1}\mathbf{p}_j^D + \mathbf{z}'_{iP2}\mathbf{p}_i^S + \mathbf{z}'_{jD1}\mathbf{x}_j^D + \mathbf{z}'_{iD2}\mathbf{x}_i^S + \mathbf{z}'_{ijD3}\mathbf{x}_{ij}\} \tag{105}$$

		Dual variables	
subject to,	$\mathbf{x}_j^D + \mathbf{z}_{jP1} = \sum_{i=1}^{R} \mathbf{x}_{ij}$,	$\mathbf{p}_j^D \geq \mathbf{0}$	(106)
	$\sum_{j=1}^{R} \mathbf{x}_{ij} + \mathbf{z}_{iP2} = \mathbf{x}_i^S$,	$\mathbf{p}_i^S \geq \mathbf{0}$	(107)
	$(\mathbf{a}_j + \hat{\mathbf{u}}_j) - (\mathbf{D}_j + \hat{\mathbf{W}}_j)\mathbf{x}_j^D + \mathbf{z}_{jD1} = \mathbf{p}_j^D$,	$\mathbf{x}_j^D \geq \mathbf{0}$	(108)
	$\mathbf{p}_i^S + \mathbf{z}_{iD2} = (\mathbf{b}_i + \hat{\mathbf{v}}_i) + (\mathbf{S}_i + \hat{\mathbf{Y}}_i)\mathbf{x}_i^S$,	$\mathbf{x}_i^S \geq \mathbf{0}$	(109)

$$\mathbf{p}_j^D + \mathbf{z}_{ijD3} = \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \hat{\boldsymbol{\lambda}}_{ij}), \qquad \mathbf{x}_{ij} \geq \mathbf{0}. \tag{110}$$

Problem (105)-(110) calibrates the observed total demand and supply quantities, $\overline{\mathbf{x}}_j^D$, $\overline{\mathbf{x}}_j^S$ and – when realized trade flows, total demands and total supplies, supply and demand prices are used as initial values in the search by the solver of the equilibrium solution – the realized trade flows, $\overline{\mathbf{x}}_{ij}$.

Adjustment vectors $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{v}}_i$ and matrices $\hat{\mathbf{W}}_j$ and $\hat{\mathbf{Y}}_i$ constitute an over-parameterization of the model. In general, therefore, it is sufficient to adjust the demand and supply functions either by modifying the corresponding intercepts or slopes.

3. Numerical Examples and Empirical Implementation

Seven numerical examples of increasing complexity will illustrate the application of the PMP methodology developed in previous sections to mathematical programming spatial trade models. The list of models is given as follows:

1. Four exporting countries and four distinct importing countries of a single commodity; unit transaction costs are measured with imprecision.
2. Four countries that are potentially export or import traders of a single commodity; unit transaction costs are measured with imprecision.
3. Four countries that are potentially export or import traders of three commodities; matrices of demand and supply slopes are diagonal, unit transaction costs are measured with imprecision.
4. Four countries that are potentially export or import traders of three commodities; full, symmetric positive semidefinite demand and supply slope matrices, unit transaction costs are measured with imprecision.
5. Four countries that are potentially export or import traders of three commodities; full, symmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured with imprecision at the same market level, unit transaction costs are measured with imprecision.
6. Four countries that are potentially export or import traders of three commodities; full, asymmetric positive semidefinite demand and supply slope matrices; unit transaction costs are measured with imprecision.
7. Four countries that are potentially export or import traders of three commodities; full, asymmetric positive semidefinite demand and supply slope matrices; demand and supply functions are measured with imprecision at the same market level; unit transaction costs are measured with imprecision.

The matrix of transaction costs may be regarded as the array of effective marginal transaction costs between trading countries with the following structure

$$\mathbf{TC} = \left[tc_{ij} + \lambda^*_{ij} \right] \tag{111}$$

where tc_{ij} is the accounting transaction cost generally measured with imprecision, and λ^*_{ij} is the differential between the effective and the accounting marginal transaction cost implied by the observed trade flows. As discussed above, in this paper, and contrary to the traditional PMP literature, the calibrating constraints are stated as a set of equations, rather than inequalities, thus the sign of λ^*_{ij} is *a priori* undetermined. This choice is based on the consideration that, if the accounting transaction costs are measured incorrectly, they may be either over or under estimated. Thus, the value and sign of the estimated λ^*_{ij} will determine the effective marginal transaction costs that will produce a calibrated solution of the quantities produced and consumed in each country.

In general, a meaningful effective transaction cost will be nonnegative. However, when trade policies are not explicitly modeled, effective transaction costs will include their effects; when export subsidies are larger than the sum of the other transaction costs, the overall effective transaction cost will be negative.

Example 1: Four exporting countries and four distinct importing countries of a single commodity; unit transaction costs are measured with imprecision.

Four countries, $I = A, B, U, E$ produce and export a single homogeneous commodity which is imported and consumed by four countries, $J = DA, DB, DU, DE$. The required information is as follows:

Parameters of the inverse demand functions:

$$\mathbf{a} = \begin{array}{c} \text{DA} \\ \text{DB} \\ \text{DU} \\ \text{DE} \end{array} \begin{bmatrix} 30.0 \\ 22.0 \\ 25.0 \\ 29.0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.55 & & & \\ & 0.37 & & \\ & & 0.42 & \\ & & & 0.49 \end{bmatrix};$$

Parameters of the inverse supply functions:

$$\mathbf{b} = \begin{array}{c} \text{A} \\ \text{B} \\ \text{U} \\ \text{E} \end{array} \begin{bmatrix} 0.4 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1.4 & & & \\ & 2.4 & & \\ & & 1.9 & \\ & & & 0.6 \end{bmatrix}$$

Matrix of accounting (observed) transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 1.2 & 1.5 & 1.0 & 0.1 \\ 1.0 & 1.0 & 0.4 & 0.5 \\ 2.0 & 0.5 & 1.5 & 2.1 \\ 3.0 & 1.2 & 2.0 & 1.0 \end{bmatrix} \end{array}$$

The optimal solution obtained without calibrating the model is as shown below:

Optimal trade flow matrix:

$$\mathbf{X}^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 13.394 & & & \\ 3.318 & & 4.662 & \\ 0.832 & 8.511 & & \\ & & 7.836 & 20.916 \end{bmatrix} \end{array}$$

Total supply quantities:

$$\mathbf{x}^{S*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[13.394,\ 7.980,\ 9.343,\ 28.752]}$$

Total demand quantities:

$$\mathbf{x}^{D*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[17.543,\ 8.511,\ 12.497,\ 20.916]}$$

Corresponding supply prices:

$$\mathbf{p}^{S*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[19.151,\ 19.351,\ 18.351,\ 17.751]}$$

and corresponding demand prices:

$$\mathbf{p}^{D*} = \underset{\quad DA \qquad DB \qquad DU \qquad DE}{[20.351,\ 18.851,\ 19.751,\ 18.751]}\ .$$

Let us now consider a matrix of realized trade flows (that differs from the above optimal matrix of trade flows):

$$\overline{\mathbf{X}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 11.500 & & 2.000 & \\ 2.500 & & 3.500 & \\ 2.000 & 7.000 & & \\ & & 6.000 & 22.500 \end{bmatrix} \end{array}$$

and the corresponding value of realized produced and consumed quantities in the eight countries considered:

$$\bar{\mathbf{x}}^S = \begin{bmatrix} 13.500, & 6.000, & 9.000, & 28.500 \end{bmatrix}$$
$$\quad A \qquad B \qquad U \qquad E$$

$$\bar{\mathbf{x}}^D = \begin{bmatrix} 16.000, & 7.000, & 11.500, & 22.500 \end{bmatrix}$$
$$\quad DA \qquad DB \qquad DU \qquad DE$$

The matrix of dual variables, Λ^*, associated with the calibrating constraints is:

$$\Lambda^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 0.700 & & -0.130 & \\ 5.600 & 3.810 & 5.170 & 2.875 \\ 1.500 & 1.210 & 0.970 & \\ 0.600 & 0.610 & 0.570 & -0.625 \end{bmatrix} \end{array}$$

And the matrix of effective unit transaction costs, $\mathbf{TC} + \Lambda^*$,:

$$\mathbf{TC} + \Lambda^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 1.900 & 1.500 & 0.870 & 0.100 \\ 6.600 & 4.810 & 5.570 & 3.375 \\ 3.500 & 1.710 & 2.470 & 2.100 \\ 3.600 & 1.810 & 2.570 & 0.375 \end{bmatrix} \end{array}.$$

The optimal solution obtained using the PMP approach, i.e. after replacing the original transaction costs with **TC** + Λ^* is as shown below:

Total supply quantities:

$$\mathbf{x}^{S*} = \begin{bmatrix} 13.500, & 6.000, & 9.000, & 28.500 \end{bmatrix}$$
$$\quad A \qquad B \qquad U \qquad E$$

Total demand quantities:

$$\mathbf{x}^{D*} = \begin{bmatrix} 16.000, & 7.000, & 11.500, & 22.500 \end{bmatrix}$$
$$\quad DA \qquad DB \qquad DU \qquad DE$$

Supply prices:

$$\mathbf{p}^{S*} = \begin{bmatrix} 19.300, & 14.600, & 17.700, & 17.600 \end{bmatrix}$$
$$\quad A \qquad B \qquad U \qquad E$$

Demand prices:

$$\mathbf{p}^{D*} = \underset{\quad DA \qquad\quad DB \qquad\quad DU \qquad\quad DE}{\left[21.200,\ 19.410,\ 20.170,\ 17.975\right]}.$$

The model calibrates exactly each country's total observed production and consumption. Multiple sets of optimal trade flows are associated to this calibration. When realized trade flows, $\bar{x}_{ij}$, are used as initial values in the optimization procedure, the optimal solution calibrates them as well.

Matrix of trade flows N.1- obtained by using realized trade flows as initial values:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 11.500 & & 2.000 & \\ 2.500 & & 3.500 & \\ 2.000 & 7.000 & & \\ & & 6.000 & 22.500 \end{bmatrix} \\ \begin{bmatrix} 16.000 & 7.000 & 11.500 & 22.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 13.500 \\ 6.000 \\ 9.000 \\ 28.500 \end{bmatrix} \\ \\ \end{array}$$

Matrix of trade flows N.2 – obtained by using alternative initial values, $\bar{x}_{ij} = 10$:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 10.327 & & 3.173 & \\ & & & 6.000 \\ 0.673 & & 8.327 & \\ 5.000 & 7.000 & & 16.500 \end{bmatrix} \\ \begin{bmatrix} 16.000 & 7.000 & 11.500 & 22.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 13.500 \\ 6.000 \\ 9.000 \\ 28.500 \end{bmatrix} \\ \\ \end{array}$$

The value of total transaction costs, $\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^*)\mathbf{x}_{ij}$, is the same in both cases and is equal to 102.412.

Example 2: Four countries that are potentially export or import traders of a single commodity; unit transaction costs are measured with imprecision.

Four countries, $R = A, B, U, E$, can potentially be either export or import traders of a single homogeneous commodity. Each country produces and consumes that commodity. The required information is as follows:

Parameters of the inverse demand functions:

$$\mathbf{a}=\begin{array}{c} A \\ B \\ U \\ E \end{array}\begin{bmatrix} 35.0 \\ 59.0 \\ 36.0 \\ 38.0 \end{bmatrix}, \qquad \mathbf{D}=\begin{bmatrix} 1.2 & & & \\ & 1.4 & & \\ & & 1.1 & \\ & & & 0.9 \end{bmatrix}$$

Parameters of the inverse supply functions:

$$\mathbf{b}=\begin{array}{c} A \\ B \\ U \\ E \end{array}\begin{bmatrix} 0.4 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}, \qquad \mathbf{S}=\begin{bmatrix} 1.4 & & & \\ & 2.4 & & \\ & & 1.9 & \\ & & & 0.6 \end{bmatrix}$$

Matrix of accounting (observed) transaction costs:

$$\mathbf{TC}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 0.10 & 4.50 & 7.50 & 9.00 \\ 4.50 & 0.10 & 7.50 & 12.00 \\ 7.50 & 7.50 & 0.10 & 7.50 \\ 9.00 & 12.00 & 7.50 & 0.10 \end{bmatrix} \end{array}$$

Without calibrating constraints – that is, computing the equilibrium solution without using the PMP approach – the optimal solution corresponds to:

Optimal trade flow matrix:

$$\mathbf{X}^{*}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.179 & 7.596 & & \\ & 11.702 & & \\ & & 11.767 & \\ & 2.570 & & 23.905 \end{bmatrix} \end{array}$$

Total supply quantities:

$$\mathbf{x}^{S*}=\begin{array}{cccc} [16.775, & 11.702, & 11.767, & 26.475] \\ A & B & U & E \end{array}$$

Total demand quantities:

$$\mathbf{x}^{D*}=\begin{array}{cccc} [9.179, & 21.868, & 11.767, & 23.905] \\ A & B & U & E \end{array}$$

Corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{cccc} [23.885, & 28.285, & 22.957, & 16.385] \\ A & B & U & E \end{array}$$

Corresponding demand prices:

$$\mathbf{p}^{D*} = \begin{array}{cccc} [23.985, & 28.385, & 23.057, & 16.485] \\ A & B & U & E \end{array}$$

Notice that supply prices differ from demand prices in each country by the amount of domestic (internal to each country) transaction cost that was specified in the amount of 0.10 for every country in the $\mathbf{TC}$ matrix. This implies that demands functions may be measured at retail level while supply functions may be measured at farm or some other intermediate level.

Let us now consider the following matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.000 & 6.000 & & \\ & 9.000 & 1.000 & \\ & 1.000 & 8.500 & 0.500 \\ 1.000 & 3.000 & & 21.000 \end{bmatrix} \end{array}$$

and the corresponding values of realized produced and consumed quantities in the four countries considered (supplies are sums over columns, demands are sums over rows of the $\bar{\mathbf{X}}$ matrix:

$$\bar{\mathbf{x}}^{S} = \begin{array}{cccc} [15.000, & 10.000, & 10.000, & 25.000] \\ A & B & U & E \end{array}$$

$$\bar{\mathbf{x}}^{D} = \begin{array}{cccc} [10.000, & 19.000, & 9.500, & 21.500] \\ A & B & U & E \end{array}$$

The matrix of dual variables, $\mathbf{\Lambda}^*$, associated with the calibrating constraints is:

$$\mathbf{\Lambda}^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 1.500 & 6.500 & & \\ & 8.100 & -6.150 & \\ & 5.300 & 5.850 & -8.450 \\ -1.500 & 4.900 & 2.550 & 3.050 \end{bmatrix} \end{array}$$

And the matrix of effective unit transaction costs, $\mathbf{TC}+\Lambda^*$:

$$\mathbf{TC}+\Lambda^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 1.600 & 11.000 & 7.500 & 9.000 \\ 4.500 & 8.200 & 1.350 & 12.000 \\ 7.500 & 12.800 & 5.950 & -0.950 \\ 7.500 & 16.900 & 10.050 & 3.150 \end{bmatrix} \end{array}$$

In general, a meaningful effective transaction cost will be nonnegative. However, when trade policies are not explicitly modeled, effective transaction costs will include the effects of missing policy instruments; for example, when export subsidies are larger than the sum of other transaction costs, the overall effective transaction cost between two countries may be negative, as is the case for one of the elements of the $\mathbf{TC}+\Lambda^*$ matrix above.
Using the calibrating constraints – that is, using the PMP approach – the optimal solution is given as:

Total supply quantities:

$$\mathbf{x}^{S*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[15.000,\ 10.000,\ 10.000,\ 25.000]}$$

Total demand quantities:

$$\mathbf{x}^{D*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[10.000,\ 19.000,\ 9.500,\ 21.500]}$$

Supply prices:

$$\mathbf{p}^{S*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[21.400,\ 24.200,\ 19.600,\ 15.500]}$$

Demand prices:

$$\mathbf{p}^{D*} = \underset{\quad A \qquad\quad B \qquad\quad U \qquad\quad E}{[23.000,\ 32.400,\ 25.550,\ 18.650]}\ .$$

In each country, supply prices are not equal to demand prices and, in particular, they differ by an amount that is much larger than the domestic transaction cost of 0.10 characterizing the price difference in the model without calibrating constraints. The amount by which supply and demand prices differ in each country is equal to the domestic effective transaction costs reported on the main diagonal of matrix ($\mathbf{TC}+\Lambda^*$). The PMP model calibrates exactly each country's total observed production and consumption, as given by the marginal sums of columns and rows of the realized matrix of trade flows. Multiple sets of optimal trade flows are associated to this calibration. When realized trade flows, $\bar{x}_{ij}$, are used as initial values in the optimization procedure, the optimal solution calibrates them as well.

Matrix of trade flows N. 1 – obtained by using realized trade flows as initial values:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.000 & 6.000 & & \\ & 9.000 & 1.000 & \\ & 1.000 & 8.500 & 0.500 \\ 1.000 & 3.000 & & 21.000 \end{bmatrix} \\ \begin{bmatrix} 10.000 & 19.000 & 9.500 & 21.500 \end{bmatrix} \end{array} \begin{bmatrix} 15.000 \\ 10.000 \\ 10.000 \\ 25.000 \end{bmatrix}$$

Matrix of trade flows N. 2 – obtained by using alternative initial values, $x_{ij} = 10$:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 10.000 & 5.000 & & \\ & 0.500 & 9.500 & \\ & & & 10.000 \\ & 13.500 & & 11.500 \end{bmatrix} \\ \begin{bmatrix} 10.000 & 19.000 & 9.500 & 21.500 \end{bmatrix} \end{array} \begin{bmatrix} 15.000 \\ 10.000 \\ 10.000 \\ 25.000 \end{bmatrix}$$

The value of total transaction costs, $\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{t}_{ij} + \boldsymbol{\lambda}_{ij}^{*})\mathbf{x}_{ij}$, is the same in both cases and equal to 342.800.

Example 3: Four countries that are potentially export or import traders of three commodities; matrices of demand and supply slopes are diagonal, unit transaction costs are measured with imprecision.

Four countries $R = A, B, U, E$ are either import or export traders of three commodities $M = 1, 2, 3$. We assume that no linkages exist across commodities either in production or consumption, that is, the matrices of the demand and supply slopes are diagonal. This means that solving this problem is analogous to solving the three individual commodity models individually. The relevant data are as follows:

Matrix of inverse demand and supply intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} \quad \mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array}$$

Matrix of inverse demand and supply slopes:

$$\mathbf{D} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} & \begin{bmatrix} 1.2 & & \\ & 2.1 & \\ & & 0.7 \\ 0.8 & & \\ & 1.6 & \\ & & 2.6 \\ 0.8 & & \\ & 0.9 & \\ & & 1.7 \\ 1.1 & & \\ & 0.8 & \\ & & 0.9 \end{bmatrix} \end{array} \quad \mathbf{S} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} & \begin{bmatrix} 1.4 & & \\ & 2.1 & \\ & & 1.7 \\ 2.4 & & \\ & 1.6 & \\ & & 1.8 \\ 1.9 & & \\ & 2.8 & \\ & & 2.1 \\ 0.6 & & \\ & 1.1 & \\ & & 0.5 \end{bmatrix} \end{array}$$

Matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.5 & 0.5 & 0.5 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.5 & 0.5 & 0.5 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

The optimal solution obtained without calibrating the model, is as shown below:

Equilibrium trade flow matrix:

$$\mathbf{X}^* = \begin{array}{cc} & \begin{array}{ccc} \quad 1 & \quad\quad 2 & \quad\quad 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} & \begin{bmatrix} 5.543 & 4.492 & 5.487 \\ 1.020 & & \\ 3.553 & & \\ & & 2.744 \\ 6.401 & 5.594 & 2.105 \\ & 2.635 & 0.038 \\ 8.244 & 0.519 & 4.690 \\ 6.904 & & 5.264 \\ 14.034 & 11.105 & 12.192 \end{bmatrix} \end{array}$$

Equilibrium total supply and demand quantities:

$$\mathbf{x}^{S*} = \begin{array}{cc} & \begin{array}{ccc} \quad 1 & \quad\quad 2 & \quad\quad 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 10.116 & 4.492 & 5.487 \\ 6.401 & 5.594 & 4.849 \\ 8.244 & 3.155 & 4.727 \\ 20.938 & 11.105 & 17.455 \end{bmatrix} \end{array} \qquad \mathbf{x}^{D*} = \begin{array}{cc} & \begin{array}{ccc} \quad 1 & \quad\quad 2 & \quad\quad 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 12.448 & 7.127 & 13.532 \\ 7.421 & 5.594 & 2.105 \\ 11.796 & 0.519 & 4.690 \\ 14.034 & 11.105 & 12.192 \end{bmatrix} \end{array}$$

Corresponding supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{cc} & \begin{array}{ccc} \quad 1 & \quad\quad 2 & \quad\quad 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 14.563 & 9.533 & 10.028 \\ 15.563 & 8.550 & 9.028 \\ 15.063 & 9.033 & 9.528 \\ 12.063 & 10.616 & 7.528 \end{bmatrix} \end{array} \qquad \mathbf{p}^{D*} = \begin{array}{cc} & \begin{array}{ccc} \quad 1 & \quad\quad 2 & \quad\quad 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 15.063 & 10.033 & 10.528 \\ 16.063 & 9.050 & 9.528 \\ 15.563 & 9.533 & 10.028 \\ 12.563 & 11.116 & 8.028 \end{bmatrix} \end{array}$$

Supply prices differ from demand prices, in each country, by the amount of 0.5 which is the domestic transaction cost as exhibited by the matrix $\mathbf{TC}$. Let us now consider the following matrix of realized trade flows:

$$
\bar{\mathbf{X}} = \begin{array}{rc}
 & \begin{array}{ccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; \end{array} \\
\begin{array}{r} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} &
\begin{bmatrix}
5.000 & 4.000 & 6.000 \\
1.000 & & \\
3.000 & & \\
1.000 & & 2.000 \\
5.000 & 5.000 & 2.000 \\
 & 1.000 & \\
 & 2.000 & \\
7.000 & & 2.500 \\
 & 1.500 & \\
6.000 & & 4.500 \\
1.000 & & 0.500 \\
12.000 & 8.000 & 10.500
\end{bmatrix}
\end{array}
$$

and the corresponding value of realized produced and consumed quantities of the three products in the four countries (sums over columns and over rows of the $\bar{\mathbf{X}}$ matrix:

$$
\bar{\mathbf{x}}^{S} = \begin{array}{rc}
 & \begin{array}{ccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; \end{array} \\
\begin{array}{r} A \\ B \\ U \\ E \end{array} &
\begin{bmatrix}
9.000 & 4.000 & 6.000 \\
6.000 & 6.000 & 4.000 \\
7.000 & 3.500 & 2.500 \\
19.000 & 8.000 & 15.500
\end{bmatrix}
\end{array}
\qquad
\bar{\mathbf{x}}^{D} = \begin{array}{rc}
 & \begin{array}{ccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; \end{array} \\
\begin{array}{r} A \\ B \\ U \\ E \end{array} &
\begin{bmatrix}
12.000 & 6.000 & 12.500 \\
7.000 & 5.000 & 2.500 \\
10.000 & & 2.500 \\
12.000 & 10.500 & 10.500
\end{bmatrix}
\end{array}
$$

When the calibrating constraints (phase I) are included in the model, the matrix of dual variables $\mathbf{\Lambda}^{*}$ (adjustment to accounting costs) is given below. Many elements of $\mathbf{\Lambda}^{*}$ are negative. However, the elements of the matrix of effective transaction costs, $\mathbf{TC}+\mathbf{\Lambda}^{*}$, are all not negative:

$$
\Lambda^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
2.100 & 3.400 & -0.150 \\
1.900 & & \\
3.000 & 0.500 & 1.850 \\
 & 0.100 & \\
-0.500 & 1.700 & 2.250 \\
1.300 & 0.300 & 0.500 \\
0.200 & & 4.050 \\
 & -1.600 & \\
1.900 & 1.400 & 5.400 \\
1.500 & & 1.450 \\
3.800 & -0.500 & 8.400 \\
 & -2.100 & 1.000 \\
1.700 & 2.200 & 1.700 \\
1.500 & & -2.050 \\
2.400 & & 3.500 \\
3.400 & 3.900 & 2.500
\end{bmatrix} \end{array}
\qquad
\mathbf{TC} + \Lambda^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
2.600 & 3.900 & 0.350 \\
3.400 & 1.500 & 1.500 \\
4.000 & 1.500 & 2.850 \\
3.000 & 3.100 & 3.000 \\
1.000 & 3.200 & 3.750 \\
1.800 & 0.800 & 1.000 \\
2.400 & 2.200 & 6.250 \\
4.000 & 2.400 & 4.000 \\
2.900 & 2.400 & 6.400 \\
3.700 & 2.200 & 3.650 \\
4.300 & 0.000 & 8.900 \\
3.700 & 1.600 & 4.700 \\
4.700 & 5.200 & 4.700 \\
5.500 & 4.000 & 1.950 \\
6.100 & 3.700 & 7.200 \\
3.900 & 4.400 & 3.000
\end{bmatrix} \end{array}
$$

.

Phase II equilibrium matrices of supply and demand quantities:

$$
\mathbf{x}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
9.000 & 4.000 & 6.000 \\
6.000 & 6.000 & 4.000 \\
7.000 & 3.500 & 2.500 \\
19.000 & 8.000 & 15.500
\end{bmatrix} \end{array}
\qquad
\mathbf{x}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
12.000 & 6.000 & 12.500 \\
7.000 & 5.000 & 2.500 \\
10.000 & & 2.500 \\
12.000 & 10.500 & 10.500
\end{bmatrix} \end{array}
$$

These matrices match the realized matrices of total demand and supply quantities.

Phase II equilibrium matrices of supply and demand prices:

$$
\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
13.000 & 8.500 & 10.900 \\
14.600 & 9.200 & 7.500 \\
12.700 & 10.000 & 4.850 \\
10.900 & 7.200 & 6.550
\end{bmatrix} \end{array}
\qquad
\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
15.600 & 12.400 & 11.250 \\
16.400 & 10.000 & 8.500 \\
17.000 & 10.000 & 13.750 \\
14.800 & 11.600 & 9.550
\end{bmatrix} \end{array}
$$

.

As previously supply and demand prices differ, in each country, by the domestic effective transaction costs exhibited in the ($\mathbf{TC} + \boldsymbol{\Lambda}^*$) matrix. The phase II equilibrium model calibrates exactly the realized trade flows as long as all the realized trade flows, and the corresponding marginal quantities of supply and demand, and demand and supply prices are used as initial values to guide the solver in search of an equilibrium solution:

Matrix of trade flows N.1 – realized trade flows are used as initial values:

$$\mathbf{X}_1^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5.000 & 4.000 & 6.000 \\ 1.000 & & \\ 3.000 & & \\ 1.000 & & 2.000 \\ 5.000 & 5.000 & 2.000 \\ & 1.000 & \\ & 2.000 & \\ 7.000 & & 2.500 \\ & 1.500 & \\ 6.000 & & 4.500 \\ 1.000 & & 0.500 \\ 12.000 & 8.000 & 10.500 \end{bmatrix} \end{array}$$

Matrix of trade flows N. 2 – alternative initial values, $\mathbf{x}_{ij} = 10$:

$$\mathbf{X}_2^* = \begin{array}{r} \\ A.A \\ A.U \\ A.E \\ B.A \\ B.B \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & 3.985 & 3.500 \\ 9.000 & & 2.500 \\ & 0.015 & \\ 6.000 & 1.000 & 4.000 \\ & 5.000 & \\ 6.000 & 1.015 & 2.000 \\ 1.000 & & \\ & 2.485 & 0.500 \\ & 0.000 & 3.000 \\ 7.000 & & 2.500 \\ 12.000 & 8.000 & 10.000 \end{bmatrix} \end{array}$$

In both equilibrium trade flows matrices, the total supplies and demands of every commodity in each country, are equal to the corresponding observed quantities. The value of total transaction costs is the same in both cases and is equal to 300.875.

Example 4: Four countries that are potentially export or import traders of three commodities; full, symmetric positive semidefinite demand and supply slope matrices, unit transaction costs are measured with imprecision.

When the Jacobian matrices of first derivatives (slope) are symmetric, systems can be integrated into a meaningful STJ objective function. The relevant data are as follows:

Matrices of inverse demand and supply intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} \qquad \mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array}$$

Matrices of inverse demand and supply slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.3 & -0.2 \\ 0.3 & 2.1 & 0.1 \\ -0.2 & 0.1 & 0.7 \\ 0.8 & -0.2 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.2 & 0.4 & 2.6 \\ 0.8 & 0.3 & 0.4 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & -0.1 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.2 & 0.9 \end{bmatrix} \end{array} \qquad \mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.4 & 2.1 & 0.2 \\ 0.3 & 0.2 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.5 & 1.6 & 0.3 \\ 0.2 & 0.3 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.5 & 0.4 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.2 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \left[\begin{array}{ccc} 0.5 & 0.5 & 0.5 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.5 & 0.5 & 0.5 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.5 & 0.5 & 0.5 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.5 & 0.5 & 0.5 \end{array}\right] \end{array}.$$

The optimal solution obtained without calibrating the model is as shown below:

Equilibrium trade flow matrix:

$$\mathbf{X}^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} & \left[\begin{array}{ccc} 3.520 & 2.764 & 4.398 \\ 3.478 & 2.787 & \\ 3.866 & & \\ & & 5.083 \\ 5.321 & 3.442 & \\ & 2.453 & 3.140 \\ 7.481 & & 0.538 \\ 9.662 & & 0.332 \\ 12.128 & 10.407 & 3.037 \end{array}\right] \end{array}$$

Equilibrium total supply and demand quantities:

$$
\mathbf{X}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 10.843 & 5.552 & 4.398 \\ 5.321 & 3.442 & 5.083 \\ 7.481 & 2.453 & 3.678 \\ 21.790 & 10.407 & 3.370 \end{bmatrix} \end{array} \quad \mathbf{X}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 13.182 & 5.218 & 12.954 \\ 8.798 & 6.229 & \\ 11.347 & & 0.538 \\ 12.128 & 10.407 & 3.037 \end{bmatrix} \end{array}
$$

Corresponding supply and demand prices:

$$
\mathbf{p}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 14.707 & 8.293 & 12.547 \\ 15.707 & 9.293 & 11.547 \\ 15.207 & 7.793 & 12.047 \\ 12.207 & 9.354 & 10.047 \end{bmatrix} \end{array} \quad \mathbf{p}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 15.207 & 8.793 & 13.047 \\ 16.207 & 9.793 & 10.749 \\ 15.707 & 6.650 & 12.547 \\ 12.707 & 9.854 & 10.547 \end{bmatrix} \end{array}
$$

Supply prices differ from demand prices, in each country, by the amount of 0.5 which is the domestic transaction cost as exhibited by the matrix $\mathbf{TC}$. Let us now consider the following matrix of realized trade flows:

$$
\bar{\mathbf{X}} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.E \end{array} & \begin{bmatrix} 3.000 & 2.500 & 4.500 \\ 2.500 & 2.000 & \\ 4.000 & & \\ & 0.500 & 4.000 \\ 2.500 & 3.500 & \\ 0.500 & & \\ 1.000 & 1.500 & 2.000 \\ 6.000 & 0.500 & \\ & & 1.000 \\ 7.000 & & 0.500 \\ 8.500 & 10.000 & 3.500 \end{bmatrix} \end{array}
$$

and the corresponding value of realized produced and consumed quantities of the three products in the four countries considered (sums over columns and over rows of the $\bar{\mathbf{X}}$ matrix):

$$
\begin{array}{c}
 \\ \overline{\mathbf{x}}^S =
\end{array}
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
A & 9.500 & 4.500 & 4.500 \\
B & 3.000 & 4.000 & 4.000 \\
U & 7.000 & 2.000 & 3.000 \\
E & 15.500 & 10.000 & 4.000
\end{array}
\qquad
\overline{\mathbf{x}}^D =
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
A & 11.000 & 4.500 & 11.000 \\
B & 5.000 & 5.500 & \\
U & 10.000 & 0.500 & \\
E & 9.000 & 10.000 & 4.500
\end{array}
$$

When the calibrating constraints are included in the model, the matrix of dual variables, Λ^* (adjustment to accounting costs) is given below. Its elements are positive and negative while the elements of the matrix of effective transaction costs are all positive:

$$
\Lambda^* =
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
A.A & 3.900 & 4.000 & 1.450 \\
A.B & 4.350 & 2.050 & \\
A.U & 2.600 & & 0.950 \\
A.E & & 0.550 & \\
B.A & 5.950 & 0.950 & 3.250 \\
B.B & 8.400 & 1.000 & 2.000 \\
B.U & 4.450 & & 2.550 \\
B.E & 1.550 & & \\
U.A & 2.650 & 3.850 & 2.850 \\
U.B & 2.900 & 1.700 & \\
U.U & 2.350 & -0.250 & 3.350 \\
U.E & & 0.200 & -3.650 \\
E.A & 6.050 & & 2.150 \\
E.B & 6.500 & & \\
E.U & 4.550 & & 1.450 \\
E.E & 6.650 & -0.150 & 0.850
\end{array}
\qquad
\mathbf{TC} + \Lambda^* =
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
A.A & 4.400 & 4.500 & 1.950 \\
A.B & 5.850 & 3.550 & 1.500 \\
A.U & 3.600 & 1.000 & 1.950 \\
A.E & 3.000 & 3.550 & 3.000 \\
B.A & 7.450 & 2.450 & 4.750 \\
B.B & 8.900 & 1.500 & 2.500 \\
B.U & 6.650 & 2.200 & 4.750 \\
B.E & 5.550 & 4.000 & 4.000 \\
U.A & 3.650 & 4.850 & 3.850 \\
U.B & 5.100 & 3.900 & 2.200 \\
U.U & 2.850 & 0.250 & 3.850 \\
U.E & 3.700 & 3.900 & 0.050 \\
E.A & 9.050 & 3.000 & 5.150 \\
E.B & 10.500 & 4.000 & 4.000 \\
E.U & 8.250 & 3.700 & 5.150 \\
E.E & 7.150 & 0.350 & 1.350
\end{array}
$$

With the calibrating constraints – that is using the PMP approach – the optimal solution is as shown below:

Total supply and demand quantities:

$$\mathbf{X}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 9.500 & 4.500 & 4.500 \\ 3.000 & 4.000 & 4.000 \\ 7.000 & 2.000 & 3.000 \\ 15.500 & 10.000 & 4.000 \end{bmatrix} \end{array} \quad \mathbf{X}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 11.000 & 4.500 & 11.000 \\ 5.000 & 5.500 & \\ 10.000 & 0.500 & \\ 9.000 & 10.000 & 4.500 \end{bmatrix} \end{array};$$

Supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 13.250 & 6.650 & 12.100 \\ 10.200 & 8.700 & 9.300 \\ 14.000 & 6.300 & 10.200 \\ 8.600 & 9.850 & 8.900 \end{bmatrix} \end{array} \quad \mathbf{p}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 17.650 & 11.150 & 14.050 \\ 19.100 & 10.200 & 11.800 \\ 16.850 & 6.550 & 14.050 \\ 15.750 & 10.200 & 10.250 \end{bmatrix} \end{array}$$

The supply prices are different from the demand prices, the differences are equal to the domestic effective transaction costs reported on the main diagonal of matrix ($\mathbf{TC} + \mathbf{\Lambda}^*$). The PMP model calibrates exactly each country's total observed production and consumption, as given by the marginal sums of columns and rows of the realized matrix of trade flows. Multiple sets of optimal trade flows are associated to this calibration. When realized trade flows, $\bar{x}_{ij}$, are used as initial values in the optimization procedure, the optimal solution calibrates them as well.

Matrix of trade flows N. 1 – obtained by using realized trade flows as initial values:

$$\mathbf{X}_1^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.E \end{array} & \begin{bmatrix} 3.000 & 2.500 & 4.500 \\ 2.500 & 2.000 & \\ 4.000 & & \\ & 0.500 & 4.000 \\ 2.500 & 3.500 & \\ 0.500 & & \\ 1.000 & 1.500 & 2.000 \\ 6.000 & 0.500 & \\ & & 1.000 \\ 7.000 & & 0.500 \\ 8.500 & 10.000 & 3.500 \end{bmatrix} \end{array}$$

Matrix of trade flows N. 2 – obtained by using alternative initial values, $x_{ij} = 10$:

$$\mathbf{X}_2^* = \begin{array}{r} \\ A.A \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 9.034 & 4.500 & \\ 0.466 & & \\ 0.003 & & 4.000 \\ & 4.000 & \\ 2.997 & & \\ 1.962 & & \\ 5.000 & 1.500 & \\ 0.038 & 0.500 & \\ & & 3.000 \\ & & 2.500 \\ 9.497 & & \\ 6.003 & 10.000 & 1.500 \end{bmatrix} \end{array}$$

The value of total transaction costs, $\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^*)\mathbf{x}_{ij}$, is the same in both cases and equal to 290.675.

Example 5: Four countries that are potentially export or import traders of three commodities; full, symmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured with imprecision at the same market level; unit transaction costs are measured with imprecision.

Here the model is calibrated in order to reproduce trade patterns as well as to adjust intercepts and the slopes of demand and supply functions so that demand prices are equal to supply prices in each region. Except for the transaction costs, which have been modified to make all the domestic ones equal to zero, input data are the same as in example 4:

Matrices of inverse demand and supply intercepts:

$$\mathbf{A} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} \qquad \mathbf{B} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array}$$

Matrices of inverse demand and supply slopes:

$$
\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
1.2 & 0.3 & -0.2 \\
0.3 & 2.1 & 0.1 \\
-0.2 & 0.1 & 0.7 \\
0.8 & -0.2 & 0.2 \\
-0.2 & 1.6 & 0.4 \\
0.2 & 0.4 & 2.6 \\
0.8 & 0.3 & 0.4 \\
0.3 & 0.9 & -0.1 \\
0.4 & -0.1 & 1.7 \\
1.1 & 0.1 & 0.3 \\
0.1 & 0.8 & 0.2 \\
0.3 & 0.2 & 0.9
\end{bmatrix} \end{array}
\qquad
\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
1.4 & -0.4 & 0.3 \\
-0.4 & 2.1 & 0.2 \\
0.3 & 0.2 & 1.7 \\
2.4 & 0.5 & 0.2 \\
0.5 & 1.6 & 0.3 \\
0.2 & 0.3 & 1.8 \\
1.9 & -0.1 & 0.5 \\
-0.1 & 2.8 & 0.4 \\
0.5 & 0.4 & 2.1 \\
0.6 & -0.1 & 0.2 \\
-0.1 & 1.1 & 0.5 \\
0.2 & 0.5 & 0.5
\end{bmatrix} \end{array}
$$

Matrix of accounting transaction costs:

$$
\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 \\
1.5 & 1.5 & 1.5 \\
1.0 & 1.0 & 1.0 \\
3.0 & 3.0 & 3.0 \\
1.5 & 1.5 & 1.5 \\
0 & 0 & 0 \\
2.2 & 2.2 & 2.2 \\
4.0 & 4.0 & 4.0 \\
1.0 & 1.0 & 1.0 \\
2.2 & 2.2 & 2.2 \\
0 & 0 & 0 \\
3.7 & 3.7 & 3.7 \\
3.0 & 3.0 & 3.0 \\
4.0 & 4.0 & 4.0 \\
3.7 & 3.7 & 3.7 \\
0 & 0 & 0
\end{bmatrix} \end{array}
$$

The optimal solution obtained without calibrating the model is as shown below:

Optimal trade flow matrix:

$$\mathbf{X}^* = \begin{array}{r} \\ A.A \\ A.B \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.071 & 5.456 & 4.710 \\ & 0.110 & \\ & & 5.076 \\ 5.418 & 3.732 & \\ & & 2.826 \\ & 2.399 & \\ 7.787 & & 0.834 \\ 2.126 & & 0.177 \\ 3.535 & & \\ 3.314 & & \\ 12.609 & 10.583 & 3.289 \end{bmatrix} \end{array} ;$$

Total supply and demand quantities:

$$\mathbf{X}^{S*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.071 & 5.567 & 4.710 \\ 5.418 & 3.732 & 5.076 \\ 7.787 & 2.399 & 3.661 \\ 21.584 & 10.583 & 3.466 \end{bmatrix} \end{array} \qquad \mathbf{X}^{D*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.196 & 5.456 & 12.790 \\ 8.954 & 6.242 & \\ 11.101 & & 0.834 \\ 12.609 & 10.583 & 3.289 \end{bmatrix} \end{array}$$

Corresponding supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.085 & 8.304 & 13.141 \\ 16.085 & 9.804 & 11.641 \\ 15.785 & 7.604 & 12.141 \\ 12.085 & 9.615 & 10.141 \end{bmatrix} \end{array} \qquad \mathbf{p}^{D*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.085 & 8.304 & 13.141 \\ 16.085 & 9.804 & 10.712 \\ 15.785 & 6.753 & 12.141 \\ 12.085 & 9.615 & 10.141 \end{bmatrix} \end{array}$$

Supply prices are now equal to demand prices, in each country, because the domestic unit transaction cost is set for all commodities at a zero level. Let us now consider the following matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 1.000 & & 3.000 \\ 3.000 & 2.000 & \\ 0.500 & & 2.000 \\ & 2.000 & \\ 6.000 & 0.500 & 0.500 \\ 2.000 & & \\ 3.000 & & \\ 2.000 & 0.500 & \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array}$$

The optimal solution obtained when imposing that demand prices must be equal to supply prices and other calibrating constraints is as shown below:
Matrices of adjustments to transaction costs $\mathbf{\Lambda}^*$ and effective transaction costs $\mathbf{TC} + \mathbf{\Lambda}^*$

$$\mathbf{\Lambda}^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.B \\ A.U \\ A.E \\ B.A \\ B.U \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.B \\ E.U \end{array} & \begin{bmatrix} -2.341 & 0.107 & -2.854 \\ -0.465 & -0.195 & -0.410 \\ -4.733 & 0.467 & -3.469 \\ -0.659 & -3.107 & -0.146 \\ -0.824 & -3.002 & -0.256 \\ -4.892 & -2.140 & -3.115 \\ -1.535 & -1.805 & -1.590 \\ -3.576 & -1.398 & -4.144 \\ -5.969 & -1.038 & -4.759 \\ -1.267 & -6.467 & -2.531 \\ -3.108 & -5.860 & -4.885 \\ -1.431 & -6.362 & -2.641 \end{bmatrix} \end{array} \qquad \mathbf{TC} + \mathbf{\Lambda}^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.B \\ A.U \\ A.E \\ B.A \\ B.U \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.B \\ E.U \end{array} & \begin{bmatrix} -0.841 & 1.607 & -1.354 \\ 0.535 & 0.805 & 0.590 \\ -1.733 & 3.467 & -0.469 \\ 0.841 & -1.607 & 1.354 \\ 1.376 & -0.802 & 1.944 \\ -0.892 & 1.860 & 0.885 \\ -0.535 & -0.805 & -0.590 \\ -1.376 & 0.802 & -1.944 \\ -2.269 & 2.662 & -1.059 \\ 1.733 & -3.467 & 0.469 \\ 0.892 & -1.860 & -0.885 \\ 2.269 & -2.662 & 1.059 \end{bmatrix} \end{array}$$

The matrix of adjustment to transaction costs, $\mathbf{\Lambda}^*$, has positive and negative elements and all $\mathbf{\Lambda}^*_{i,i,k}$ are zero, as expected.

The deviations from supply and demand intercepts are given by:

$$\hat{\mathbf{V}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -0.522 & 0.174 & 0.028 \\ 0.054 & 0.116 & 0.104 \\ 0.029 & -0.043 & 0.098 \\ 0.006 & 0.005 & 0.014 \end{bmatrix} \end{array} \qquad \hat{\mathbf{U}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.522 & -0.174 & -0.028 \\ -0.054 & -0.116 & -0.104 \\ -0.029 & 0.043 & -0.098 \\ -0.006 & -0.005 & -0.014 \end{bmatrix} \end{array}$$

Matrix $\hat{\mathbf{U}}$ is the negative of matrix $\hat{\mathbf{V}}$. This is because of the over-parameterization of the model, as indicated above. The elimination of these deviation matrices does not affect the calibration of the trade model.

Deviations of supply and demand slopes are given by:

$$\hat{\mathbf{Y}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.001 & -0.025 & -0.012 \\ -0.025 & 0.611 & 0.302 \\ -0.012 & 0.302 & 0.149 \\ 0.253 & 0.255 & 0.282 \\ 0.255 & 0.257 & 0.284 \\ 0.282 & 0.284 & 0.314 \\ 0.267 & -0.012 & 0.286 \\ -0.012 & 5.2E\text{-}4 & -0.013 \\ 0.286 & -0.013 & 0.306 \\ 0.124 & 0.078 & 0.100 \\ 0.078 & 0.049 & 0.063 \\ 0.100 & 0.063 & 0.081 \end{bmatrix} \end{array} \qquad \hat{\mathbf{W}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.019 & -0.111 & -0.110 \\ -0.111 & 0.638 & 0.638 \\ -0.110 & 0.638 & 0.637 \\ 0.398 & 0.433 & 0.218 \\ 0.433 & 0.471 & 0.237 \\ 0.218 & 0.237 & 0.119 \\ 0.352 & -0.074 & 0.265 \\ -0.074 & 0.016 & -0.056 \\ 0.265 & -0.056 & 0.200 \\ 0.072 & 0.060 & 0.066 \\ 0.060 & 0.051 & 0.056 \\ 0.066 & 0.056 & 0.062 \end{bmatrix} \end{array}$$

In phase II, when the estimates of the adjustments are included in the model and calibrating constraints omitted, the optimal solution is as shown below:

Total supply and demand quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 4.000 & 2.000 & 3.000 \\ 6.500 & 2.500 & 2.500 \\ 18.000 & 9.500 & 2.000 \end{bmatrix} \end{array} \qquad \mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.500 & 8.000 \\ 6.000 & 4.000 & \\ 8.000 & 1.000 & 0.500 \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array}$$

Matrices of supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 14.665 & 6.594 & 11.197 \\ 13.824 & 8.201 & 9.844 \\ 15.200 & 7.399 & 11.787 \\ 12.931 & 10.061 & 10.728 \end{bmatrix} \end{array} \qquad \mathbf{p}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 14.665 & 6.594 & 11.197 \\ 13.824 & 8.201 & 9.844 \\ 15.200 & 7.399 & 11.787 \\ 12.931 & 10.061 & 10.728 \end{bmatrix} \end{array}$$

The model calibrates exactly each country's production and consumption of the three commodities and in each country demand prices equal supply prices. Two examples of optimal trade flows matrix associated to this optimal solution are provided below. The first one – obtained using the realized trade flows, and the corresponding total demand and supply quantities and demand and supply prices as initial values – calibrates exactly the observed trade flows The second matrix is obtained using alternative initial values, $x_{ij} = 10$, and the model does not calibrate the observed trade flows.

$$\mathbf{X}_1^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 1.000 & & 3.000 \\ 3.000 & 2.000 & \\ 0.500 & & 2.000 \\ & 2.000 & \\ 6.000 & 0.500 & 0.500 \\ 2.000 & & \\ 3.000 & & \\ 2.000 & 0.500 & \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array} \qquad \mathbf{X}_2^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 0.500 & & 1.000 \\ 6.000 & & \\ & 1.000 & \\ 4.500 & 2.500 & 2.000 \\ 4.000 & & 3.000 \\ & 2.000 & \\ & & 2.000 \\ & & 0.500 \\ 6.500 & 2.500 & \\ 10.000 & 3.500 & \\ & 4.000 & \\ 8.000 & & \\ & 2.000 & \end{bmatrix} \end{array}$$

It can be easily verified that in the three cases the model calibrates exactly on total demanded and supplied quantities in each country. The value of total transaction costs is the same in all three cases and equal to 14.406.

Example 6: Four countries that are potentially export or import traders of three commodities; full, asymmetric positive semidefinite demand and supply slope matrices; unit transaction costs are measured with imprecision.

In general, systems of demand and supply functions do not exhibit symmetric Jacobian matrices of first derivatives (slopes). When three or more commodities are involved, these systems cannot be integrated into a meaningful STJ objective function. The solution of such trade models relies upon the specification and solution of an Equilibrium Problem, as illustrated in section 2.4. The following numerical example exhibits asymmetric matrices of demand and supply slopes. The relevant data are as follows:

Matrices of inverse demand and supply intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} \qquad \mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array}$$

Matrices of inverse demand and supply slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.2 & -0.2 \\ 0.3 & 2.1 & 0.2 \\ -0.1 & 0.1 & 0.7 \\ 0.8 & -0.1 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.3 & 0.3 & 2.6 \\ 0.8 & 0.2 & 0.5 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & 0.0 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.0 & 0.8 & 0.2 \\ 0.4 & 0.3 & 0.9 \end{bmatrix} \end{array} \qquad \mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.2 & 2.1 & 0.2 \\ 0.2 & 0.3 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.7 & 1.6 & 0.3 \\ 0.1 & 0.5 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.6 & 0.5 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.3 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

Matrix of accounting transaction costs:

$$
\mathbf{TC} = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
0.5 & 0.5 & 0.5 \\
1.5 & 1.5 & 1.5 \\
1.0 & 1.0 & 1.0 \\
3.0 & 3.0 & 3.0 \\
1.5 & 1.5 & 1.5 \\
0.5 & 0.5 & 0.5 \\
2.2 & 2.2 & 2.2 \\
4.0 & 4.0 & 4.0 \\
1.0 & 1.0 & 1.0 \\
2.2 & 2.2 & 2.2 \\
0.5 & 0.5 & 0.5 \\
3.7 & 3.7 & 3.7 \\
3.0 & 3.0 & 3.0 \\
4.0 & 4.0 & 4.0 \\
3.7 & 3.7 & 3.7 \\
0.5 & 0.5 & 0.5
\end{bmatrix} \end{array}
$$

The optimal solution obtained without calibrating the model, is shown below:

Equilibrium trade flow matrix:

$$
\mathbf{X}^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
3.910 & 1.740 & 4.637 \\
2.834 & 2.887 & \\
3.684 & & \\
 & & 4.835 \\
5.356 & 3.037 & \\
 & 2.704 & 2.037 \\
7.618 & & 0.837 \\
9.809 & & 0.450 \\
12.909 & 12.124 & 0.158
\end{bmatrix} \end{array}
$$

Equilibrium total supply and demand quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.429 & 4.627 & 4.637 \\ 5.356 & 3.037 & 4.835 \\ 7.618 & 2.704 & 2.873 \\ 22.718 & 12.124 & 0.608 \end{bmatrix} \end{array} \quad \mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.719 & 4.445 & 11.958 \\ 8.190 & 5.924 & \\ 11.302 & & 0.837 \\ 12.909 & 12.124 & 0.158 \end{bmatrix} \end{array}$$

Corresponding supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.540 & 8.659 & 12.057 \\ 15.540 & 9.659 & 11.057 \\ 15.040 & 8.159 & 11.557 \\ 12.040 & 9.769 & 9.557 \end{bmatrix} \end{array} \quad \mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.040 & 9.159 & 12.557 \\ 16.040 & 10.159 & 10.766 \\ 15.540 & 8.659 & 12.057 \\ 12.540 & 10.269 & 10.057 \end{bmatrix} \end{array}$$

Supply prices differ from demand prices, in each country, by the amount of 0.5 which is the domestic transaction cost as exhibited by the matrix $\mathbf{TC}$. Let us now consider the following matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.000 & 2.000 & 3.000 \\ 2.500 & 2.500 & \\ 2.000 & & \\ 0.500 & & 4.000 \\ 5.000 & 2.000 & \\ 1.000 & 1.000 & 1.000 \\ 6.000 & & \\ 10.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array}$$

and the corresponding values of realized produced and consumed quantities of the three products in the four countries (sums over columns and over rows of the $\bar{\mathbf{X}}$ matrix):

$$\bar{\mathbf{x}}^{S} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 4.500 & 3.000 \\ 5.500 & 2.000 & 4.000 \\ 7.000 & 1.000 & 1.000 \\ 22.000 & 10.000 & \end{bmatrix} \end{array} \quad \bar{\mathbf{x}}^{D} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.000 & 8.000 \\ 7.500 & 4.500 & \\ 8.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array}$$

When the calibrating constraints (phase I) are included in the model, the matrix of dual variables Λ^* (adjustment to accounting costs) is given below. Many of its elements are negative. The same is true for the matrix of effective transaction costs $\mathbf{TC}+\Lambda^*$:

$$\Lambda^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.100 & 3.600 & 6.400 \\ 4.950 & 2.150 & 1.250 \\ 7.600 & -2.050 & 5.150 \\ 0.800 & 0.350 & -0.450 \\ -3.100 & 3.400 & 5.000 \\ 0.750 & 3.950 & 1.850 \\ 1.200 & -2.450 & 3.550 \\ -5.400 & 0.150 & -1.850 \\ -0.500 & 9.050 & 8.150 \\ 1.150 & 7.400 & 2.800 \\ 5.000 & 4.400 & 7.900 \\ -3.000 & 5.600 & 1.100 \\ -1.100 & 2.550 & 8.350 \\ 0.750 & 1.100 & 3.200 \\ 3.200 & -3.300 & 6.900 \\ 1.600 & 4.300 & 6.500 \end{bmatrix} \end{array} \qquad \mathbf{TC}+\Lambda^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.600 & 4.100 & 6.900 \\ 6.450 & 3.650 & 2.750 \\ 8.600 & -1.050 & 6.150 \\ 3.800 & 3.350 & 2.550 \\ -1.600 & 4.900 & 6.500 \\ 1.250 & 4.450 & 2.350 \\ 3.400 & -0.250 & 5.750 \\ -1.400 & 4.150 & 2.150 \\ 0.500 & 10.050 & 9.150 \\ 3.350 & 9.600 & 5.000 \\ 5.500 & 4.900 & 8.400 \\ 0.700 & 9.300 & 4.800 \\ 1.900 & 5.550 & 11.350 \\ 4.750 & 5.100 & 7.200 \\ 6.900 & 0.400 & 10.600 \\ 2.100 & 4.800 & 7.000 \end{bmatrix} \end{array}$$

Phase II equilibrium matrices of supply and demand quantities:

$$\mathbf{x}^{S*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 4.500 & 3.000 \\ 5.500 & 2.000 & 4.000 \\ 7.000 & 1.000 & 1.000 \\ 22.000 & 10.000 & \end{bmatrix} \end{array} \qquad \mathbf{x}^{D*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.000 & 8.000 \\ 7.500 & 4.500 & \\ 8.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array}$$

These matrices match the corresponding realized matrices of total demand and supply quantities.

Phase II equilibrium matrices of supply and demand prices:

$$\mathbf{p}^{S*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 8.650 & 8.650 \\ 15.200 & 7.850 & 9.050 \\ 13.100 & 2.700 & 6.400 \\ 11.700 & 7.200 & 4.200 \end{bmatrix} \end{array} \qquad \mathbf{p}^{D*} = \begin{array}{r} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.600 & 12.750 & 15.550 \\ 16.450 & 12.300 & 11.400 \\ 18.600 & 7.600 & 14.800 \\ 13.800 & 12.000 & 11.200 \end{bmatrix} \end{array}$$

These supply and demand prices differ, in each country, by the domestic effective transaction costs exhibited in the $(\mathbf{TC}+\mathbf{\Lambda}^*)$ matrix. The phase II equilibrium model calibrates exactly the realized trade flows as long as all the available information is used (as long as the realized trade flows, and the corresponding marginal quantities of supply and demand, and demand and supply prices are used as initial values) to guide the solver in search of an equilibrium solution:
Matrix of trade flows N. 1 – realized trade flows, and corresponding marginal quantities of supply and demand, and demand and supply prices are used as initial values:

$$\mathbf{X}_1^* = \begin{array}{c} \\ \begin{array}{r} A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.000 & 2.000 & 3.000 \\ 2.500 & 2.500 & \\ 2.000 & & \\ 0.500 & & 4.000 \\ 5.000 & 2.000 & \\ 1.000 & 1.000 & 1.000 \\ 6.000 & & \\ 10.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array}$$

Matrix of trade flows N. 2 – alternative initial values, $\mathbf{x}_{ij} = 10$:

$$\mathbf{X}_2^* = \begin{array}{c} \\ \begin{array}{r} A.A \\ A.B \\ B.A \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 3.000 & 3.000 \\ & 1.500 & \\ & & 4.000 \\ 5.500 & 2.000 & \\ 4.484 & & 1.000 \\ 2.516 & 0.677 & \\ & 0.323 & \\ 2.516 & & \\ 4.984 & 2.323 & \\ 8.000 & & \\ 6.500 & 7.677 & \end{bmatrix} \end{array}$$

In both equilibrium trade flow matrices, the total supplies and demands of every commodity is equal to the corresponding observed quantities. The value of total transaction costs is the same in both cases and is equal to 267.400.

Example 7: Four countries that are potentially export or import traders of three commodities; full, asymmetric positive semidefinite demand and supply slope matrices; demand and supply functions are measured at the same market level with imprecision; unit transaction costs are measured with imprecision.

The corresponding model is calibrated to reproduce observed trade patterns as well as to adjust intercepts and slopes of demand and supply functions so that demand prices will equal supply prices in each region. Except for the transaction costs (where, now, the domestic amounts are all set equal to zero), input data are the same as in example 6. The modified data are as follows:

Matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.0 & 0.0 & 0.0 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.0 & 0.0 & 0.0 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \end{array}$$

The optimal solution obtained without calibrating the model is shown below.

Equilibrium trade flows matrix:

$$
\mathbf{X}^{*} = \begin{array}{c} \\ A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
10.640 & 4.653 & 4.943 \\
 & & 4.794 \\
5.461 & 3.305 & \\
 & & 1.673 \\
 & 2.646 & \\
7.930 & & 1.159 \\
3.125 & & 0.423 \\
2.886 & & \\
3.074 & & \\
13.405 & 12.311 & 0.352
\end{bmatrix}
\end{array}
$$

Total supply and demand quantities:

$$
\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
10.640 & 4.653 & 4.943 \\
5.461 & 3.305 & 4.794 \\
7.930 & 2.646 & 2.832 \\
22.490 & 12.311 & 0.775
\end{bmatrix}
\end{array}
\qquad
\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
13.765 & 4.653 & 11.833 \\
8.347 & 5.951 & \\
11.003 & & 1.159 \\
13.405 & 12.311 & 0.352
\end{bmatrix}
\end{array}
$$

Corresponding supply and demand prices:

$$
\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
14.918 & 8.732 & 12.628 \\
15.918 & 10.148 & 11.128 \\
15.618 & 7.948 & 11.628 \\
11.918 & 10.081 & 9.628
\end{bmatrix}
\end{array}
\qquad
\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix}
14.918 & 8.732 & 12.628 \\
15.918 & 10.148 & 11.128 \\
15.618 & 7.948 & 11.628 \\
11.918 & 10.081 & 9.628
\end{bmatrix}
\end{array}.
$$

Supply prices are now equal to demand prices, in each country, because the domestic unit transaction cost is set for all commodities at a zero level. Let us now consider the following matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{r} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & \\ 0.500 & & 3.500 \\ 5.000 & 2.000 & \\ & 2.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 1.500 & \\ 1.500 & & \\ 11.000 & 11.000 & \end{bmatrix} \end{array}$$

The equilibrium solution obtained when imposing the condition that demand prices must be equal to supply prices and other calibrating constraints is shown below:
Matrix of adjustments to transaction costs $\mathbf{\Lambda}^*$ and effective transaction costs $\mathbf{TC}+\mathbf{\Lambda}^*$:

$$\mathbf{\Lambda}^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.000 & 0.000 & 0.000 \\ -2.354 & 2.569 & -2.442 \\ -1.557 & -0.224 & -0.785 \\ -3.810 & 3.123 & -1.868 \\ -0.646 & -2.569 & 0.942 \\ 0.000 & 0.000 & 0.000 \\ -0.703 & -2.793 & 1.657 \\ -2.956 & -3.446 & 0.574 \\ 1.557 & 0.224 & -0.215 \\ 0.703 & 0.593 & -1.657 \\ 0.000 & 0.000 & 0.000 \\ -2.253 & 3.347 & -1.083 \\ 0.810 & -3.123 & -1.132 \\ -1.044 & -4.554 & -0.574 \\ -1.447 & -3.347 & 1.083 \\ 0.000 & 0.000 & 0.000 \end{bmatrix} \end{array} \qquad \mathbf{TC}+\mathbf{\Lambda}^* = \begin{array}{r} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.000 & 0.000 & 0.000 \\ -0.854 & 4.069 & -0.942 \\ -0.557 & 0.776 & 0.215 \\ -0.810 & 6.123 & 1.132 \\ 0.854 & -1.069 & 2.442 \\ 0.000 & 0.000 & 0.000 \\ 1.497 & -0.593 & 3.857 \\ 1.044 & 0.554 & 4.574 \\ 2.557 & 1.224 & 0.785 \\ 2.903 & 2.793 & 0.543 \\ 0.000 & 0.000 & 0.000 \\ 1.447 & 7.047 & 2.617 \\ 3.810 & -0.123 & 1.868 \\ 2.956 & -0.554 & 3.426 \\ 2.253 & 0.353 & 4.783 \\ 0.000 & 0.000 & 0.000 \end{bmatrix} \end{array}$$

Deviations of demand and supply intercepts:

$$
\hat{\mathbf{V}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.000 & 0.031 & 0.010 \\ 0.002 & 0.028 & 0.019 \\ 0.053 & 0.014 & 0.010 \\ 0.009 & 0.001 & 0.005 \end{bmatrix} \end{array}
\qquad
\hat{\mathbf{U}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.000 & -0.031 & -0.010 \\ -0.002 & -0.028 & -0.019 \\ -0.053 & -0.014 & -0.010 \\ -0.009 & -0.001 & -0.005 \end{bmatrix} \end{array}
$$

Matrix $\hat{\mathbf{U}}$ is the negative of matrix $\hat{\mathbf{V}}$. This is because of the over-parameterization of the model, as indicated above. The elimination of these deviation matrices does not affect the calibration of the trade model.

Deviations of supply (**Y**) and demand (**W**) slopes:

$$
\hat{\boldsymbol{Y}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.008 & 0.002 & 0.003 \\ 0.342 & 0.062 & 0.125 \\ 0.115 & 0.021 & 0.042 \\ 0.012 & 0.005 & 0.007 \\ 0.154 & 0.070 & 0.098 \\ 0.104 & 0.047 & 0.066 \\ 0.318 & 0.106 & 0.159 \\ 0.084 & 0.028 & 0.042 \\ 0.059 & 0.020 & 0.030 \\ 0.151 & 0.102 & 0.009 \\ 0.024 & 0.016 & 0.001 \\ 0.083 & 0.056 & 0.005 \end{bmatrix} \end{array}
\qquad
\hat{\boldsymbol{W}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.010 & 0.002 & 0.007 \\ 0.389 & 0.062 & 0.296 \\ 0.130 & 0.021 & 0.099 \\ 0.018 & 0.009 & 0.000 \\ 0.237 & 0.126 & 0.000 \\ 0.160 & 0.085 & 0.000 \\ 0.398 & 0.000 & 0.106 \\ 0.104 & 0.000 & 0.028 \\ 0.074 & 0.000 & 0.020 \\ 0.097 & 0.102 & 0.000 \\ 0.015 & 0.016 & 0.000 \\ 0.054 & 0.056 & 0.000 \end{bmatrix} \end{array}
$$

Phase II equilibrium matrices of supply and demand quantities:

$$
\mathbf{x}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 2.000 & 4.000 \\ 5.500 & 4.500 & 3.500 \\ 6.000 & 2.000 & 3.000 \\ 17.000 & 11.500 & 1.000 \end{bmatrix} \end{array}
\qquad
\mathbf{x}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 12.500 & 2.000 & 9.500 \\ 8.500 & 4.500 & \\ 7.500 & & 2.000 \\ 11.000 & 11.500 & \end{bmatrix} \end{array}
$$

Equilibrium matrices of supply and demand prices

$$\mathbf{p}^{S*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 16.308 & 7.321 & 11.780 \\ 15.454 & 9.889 & 9.338 \\ 14.752 & 7.097 & 10.994 \\ 12.498 & 10.443 & 9.912 \end{bmatrix} \end{array} \qquad \mathbf{p}^{D*} = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A \\ B \\ U \\ E \end{array} & \begin{bmatrix} 16.308 & 7.321 & 11.780 \\ 15.454 & 9.889 & 9.338 \\ 14.752 & 7.097 & 10.994 \\ 12.498 & 10.443 & 9.912 \end{bmatrix} \end{array}$$

Supply and demand prices within each country are equal, as required by theory when demand and supply functions are measured at the same market level. Also this model calibrates exactly each country's observed production and consumption of the three commodities. This equilibrium model exhibits multiple equilibrium solutions; two examples of equilibrium sets of trade flows are shown below. The first matrix – obtained using the realized trade flows, and the corresponding total demand and supply quantities and demand and supply prices as initial values - calibrates exactly the observed trade flows. The second matrix is obtained using alternative initial values, $\mathbf{x}_{ij} = 10$, and the model does not calibrate the observed trade flows:

$$\mathbf{X}_1^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & 3.500 \\ 0.500 & & \\ 5.000 & 2.000 & \\ & 2.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 1.500 & \\ 1.500 & & \\ 11.000 & 11.000 & \end{bmatrix} \end{array} \qquad \mathbf{X}_2^* = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} & \begin{bmatrix} 9.506 & 2.000 & 4.000 \\ 1.494 & & \\ & & 3.500 \\ 5.500 & 1.828 & \\ & 0.672 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.994 & & 1.000 \\ 1.506 & 0.672 & \\ 1.500 & & \\ 11.000 & 10.828 & \end{bmatrix} \end{array}$$

In both equilibrium trade flow matrices, the total supplies and demands of every commodity is equal to the corresponding observed quantities. The value of total transaction costs is the same in both cases and is equal to 34.747.

References

Anania G. (2001). *Modeling Agricultural Trade Liberalization. A Review*. Paper presented at the annual meeting of the American Agricultural Economics Association, Chicago, August 5-8.

Armington P. A. (1969). *A theory of demand for products distinguished by place of production*. IMF Staff Papers.16.

Bauer S. and H. Kasnakoglu (1990). "Non-linear programming models for sector and policy analysis". *Economic Modelling*. July: 275-290.

Bouamra-Mechemache Z., J.-P. Chavas, T. Cox and V. Réquillart (2002). "EU Dairy Policy Reform and Future WTO Negotiations: a Spatial Equilibrium Analysis". *Journal of Agricultural Economics.* (53) 2: 233-257.

Bouët A. (2008). *The Expected Benefits of Trade Liberalization for World Income and Development. Opening the "Black Box" of Global Trade Modeling*. Food Policy Review no. 8, International Food Policy Research Institute, Washington D.C..

Dantzig G. B. (1951). "Application of the Simplex Method to a Transportation Problem". In T. C. Koopmans (ed). *Activity Analysis of Production and Allocation*. New York: John Wiley and Sons Inc.: 359-373.

Francois J. F. and K. A. Reinert (1997). *Applied Methods for Trade Policy Analysis. A Handbook*. Cambridge: Cambridge University Press.

Howitt R. (1995a). "Positive Mathematical Programming". *American Journal of Agricultural Economics*. (77): 329-342.

Howitt R. (1995b). "A Calibration Method for Agricultural Economic Production Models". *Journal of Agricultural Economics*. (46) 2: 147-159.

Jansson T. and T. Heckelei (2009). "A new estimator for trade costs and its small sample properties". *Economic Modelling*. (26): 489-498.

Koopmans T. C. (1947). "Optimum Utilization of the Transportation System". *Proceedings of the International Statistical Conference*. Washington, D.C., (5), reprinted as Supplement to *Econometrica*, (17), 1949.

Paris Q. (1981). "Multiple Optimal Solutions in Linear Programming Models". *American Journal of Agricultural Economics*. (63): 724-727.

Paris Q. (1983). "Multiple Optimal Solutions in Quadratic Programming Models". *Western Journal of Agricultural Economics*. (8): 141-154.

Samuelson P. (1952). "Spatial Price Equilibrium and Linear Programming". *American Economic Review*. (42): 283-303.

Takayama T. and G. G. Judge (1971). *Spatial and Temporal Price and Allocation Models*. North Holland, Amsterdam.

van Tongeren F., H. van Meijl and Y. Surry (2001). "Global models applied to agricultural and trade policies: a review and assessment". *Agricultural Economics*. (26): 149-172.