Positive Mathematical Programming (PMP) is an approach to empirical analysis that uses all the available information, no matter how scarce. It uses sample and user-supplied information in the form of expert opinion. This approach is especially useful in situations where only short time series are available as, for example, in sectoral analyses of developing countries and environmental economics analyses. PMP is a policy oriented approach. By this characterization we mean that, although the structure of the PMP specification assumes the form of a mathematical programming model, the ultimate objective of the analysis is to formulate policy recommendations. In this regard, PMP is not different from a traditional econometric analysis.

PMP grew out of two distinct dissatisfactions with current methodologies: First, the inability of standard econometrics to deal with limited and incomplete information and, second, the inability of linear programming (LP) to approximate even roughly realized farm production plans and, therefore, to become a useful methodology for policy analysis. The original work on PMP is due to Howitt. After the sixties, a strange idea spread like a virus among empirical economists: Only traditional econometric techniques were considered as legitimate tools for economic analysis. On the contrary, mathematical programming techniques, represented mainly by LP (and which had flourished along side traditional econometrics in the previous decade), were regarded as inadequate tools for interpreting economic behavior and for policy analysis. In reality, the emphasis on linear programming applications during the fifties and sixties provided ammunitions to the critics of mathematical programming who could not see the analytical potential of quadratic and, in general, nonlinear programming.

It is well known that LP specializes a model’s solution beyond the economic reality observed in any sample of firms. To restate the same idea using the dimensions of a LP model, the number of production activities that are operated at a positive level cannot exceed the number of con-
straints. If the constraints are only a few—as in the case of commercial agriculture—a LP model may result in a solution with even fewer production activities operated at positive levels that will be very different from those observed in reality. Furthermore, a LP model is articulated in such a manner that all the net revenue is allotted to the limiting inputs leaving a farm entrepreneur, identically and unrealistically, with zero profit.

A second methodological dissatisfaction was identified with the perceived dichotomy between econometrics and mathematical programming. During the nineteen sixties and seventies, a questionable characterization of econometrics and mathematical programming took place in the literature. Econometrics was seen as a “positive” methodology while mathematical programming was labeled as a “normative” approach.

Without entering into a deep philosophical discussion, a “positive” approach is one that observes (measures) “facts” as consequences of economic agents’ decisions and uses a methodological apparatus that avoids recommendations suggesting actions based upon verbs such as “ought to, must, should.” A “positive” approach is based on observed data while in a “normative” approach an economic agent must actively follow a recommendation in order to achieve a given objective. For example, in a traditional mathematical programming analysis that attempts to maximize an objective function subject to a set of constraints, the recommendation would be that the economic agent who desires to maximize the specified objective function ought to implement the plan resulting from the solution of the model. The “positive” aspect of econometrics, instead, could be illustrated by the fact that the main objective of the methodology deals with the utilization and analysis of information contained in a sample of data representing the realized decisions of economic agents facing specific technological, environmental and market information.

In order to illustrate the difference between the econometric approach and a traditional mathematical programming model we will study the anatomy of each stylized model in terms of the given (known) and unknown information associated with each specification. First, let us suppose that an econometrician is interested in estimating a supply function for a given commodity using a vector of sample information on output quantity, \( y \), and a matrix of output and input prices, \( X \). Let us assume also that she chooses a linear model to represent the supply function as

\[
y = X\beta + u
\]  

(14.1)

where \( \beta \) is a vector of unknown parameters that—according to the econometrician—define the structure of the supply function and \( u \) is a vector of (unknown) random disturbances. This linear model and the methodology to estimate it are called “positive” because they are based upon the sample information \((y, X)\). Little or no attention is paid to the fact that a
supply function is the result of a profit maximizing assumption requiring an implicit statement such as “in order to maximize profit, a price-taking economic agent must (ought to, should) equate the value marginal product of each input to the corresponding input price.”

On the contrary, mathematical programming is viewed as a “normative” approach because a typical model requires an explicit optimization operator such as either “maximize” or “minimize.” Let us consider the following linear programming model that refers to, say, a farmer who is a price-taking entrepreneur and wishes to maximize profit, precisely as those entrepreneurs associated with the econometric model (14.1):

\[
\begin{align*}
\text{maximize} \quad TNR &= \mathbf{p}'\mathbf{x} - \mathbf{v}'\mathbf{x} \\
\text{subject to} \quad A\mathbf{x} &\leq \mathbf{b} \\
\mathbf{x} &\geq 0.
\end{align*}
\] (14.2)

In the LP model (14.2), the vector \( \mathbf{p} \) represents known output prices, the vector \( \mathbf{v} \) represents known (variable) accounting costs per unit of output, the matrix \( A \) represents the known technology available to the price-taking entrepreneur. This technology articulates the structure of the problem analogously to the \( \beta \) vector of the econometric model (14.1) and is usually obtained in consultation with experts in the field. The vector \( \mathbf{b} \) represents the known quantities of available and limiting resources. Finally, the vector \( \mathbf{x} \) represents the unknown output levels of the profit-maximizing farmer. An example of accounting costs may be fertilizer and pesticide expenses. \( TNR \) stands for total net revenue, since \( \mathbf{p}'\mathbf{x} \) is total revenue and \( \mathbf{v}'\mathbf{x} \) is total variable cost. Hence, when mathematical programming is used to improve the production plan of a firm, the methodology can certainly be regarded as “normative.” We are interested, however, in using the methodology of mathematical programming for policy analysis. This is the reason for comparing it to the econometric methodology and integrating it in those missing parts that make econometrics a positive methodology.

The essential difference between the two models (14.1) and (14.2) is that the output levels are known in the econometric model (14.1) and are unknown in the LP model (14.2). Hence, we conclude that, if the “positive” character of the econometric model (14.1) depends upon the utilization of the sample information \( \mathbf{y} \), the same “positive” characterization will apply to the LP model (14.2) as soon as one will associate the known output levels with the structure of model (14.2).

Notice that in the econometric model (14.1) the information given by \( \mathbf{y} \) is represented by \( \mathbf{x}_{\text{obs}} \) in model (14.3). To homogenize the different notation used in the two models for indicating the same information, let \( \mathbf{x}_{\text{obs}} \equiv \mathbf{y} \) and \( \mathbf{X} \equiv (\mathbf{p}, \mathbf{v}) \) represents the information in question, where \( \mathbf{x}_{\text{obs}} \) is the “observed” level of output that may be part of the sample information. Then, the following LP model contains essentially the same information
that defines the econometric model (14.1) and, thus, can be regarded as a Positive Mathematical Programming model:

\[
\text{maximize} \quad TNR = p'x - v'x \quad \quad (14.3)
\]

subject to \quad \quad Ax \leq b, \quad y
\quad x \leq x_{obs}, \quad \quad \lambda
\quad x \geq 0.

Constraints \((Ax \leq b)\) are called structural constraints, while constraints \((x \leq x_{obs})\) are called calibrating constraints. The meaning of calibration is the adjustment of the value of a reading by comparison to a standard: the reading is the vector \(x\) and the standard is the vector \(x_{obs}\).

The dual variables \(y\) and \(\lambda\) are associated, respectively, with the limiting resources, \(b\), and the observed output levels, \(x_{obs}\). Here, the symbol \(y\) no longer represents the sample information of the econometric model but rather the vector of dual variables of limiting resources, as in previous chapters. The novel constraint \(x \leq x_{obs}\) constitutes the characterizing element of the PMP model. Any dual variable (or Lagrange multiplier) is interpreted as the marginal sacrifice associated with the corresponding constraint. Hence, the direction of the inequality is justified by the meaning of the corresponding dual variable, \(\lambda\), as the marginal cost of the corresponding observed output levels. Without the specified direction of the inequality constraint, the marginal cost would not be guaranteed to be nonnegative.

An important comment, that anticipates an expected criticism of the PMP model, regards the apparent tautology of model (14.3) where it is clear, by inspection, that an optimal solution will occur at \(x_{opt} = x_{obs}\). The expected question: What is the use of a LP model for which one knows in advance the optimal solution? The answer rests on the measurement of the marginal costs of the observed output levels, \(\lambda\). This type of information is missing in the econometric model (14.1) but it constitutes an essential piece of economic information about the behavior of any economic agent.

As to the tautology associated with the PMP model (14.3), that is \(x_{opt} = x_{obs}\), we observe that any statistical and econometric model attempts to reach the same type of tautology. In fact, it will be easily recognized that the goodness of fit of any econometric model hinges upon the residual vector \(\hat{u} = y - \hat{y}\) being as small as possible in all its components, preferably equal to zero. Hence, a successful econometrician would like to get as close as possible (albeit without ever achieving it) to a tautology similar to the tautology achieved in a PMP model.

A schematic representation of the relationship between econometric and PMP information is given in figure 14.1. Without the double arrow in the bottom part of the PMP section, the structure and the output quantity exhibits opposite known and unknown elements of the respective econo-
Econometrics and PMP specifications. It is precisely the double arrow that distinguishes traditional mathematical programming models from PMP models and confers upon them the desired “positive” character.

\[ y = \beta X + u \]
\[ \text{max} \ TNR = p'x - v'x \]
\[ \text{s.t.} \ Ax < b \]
\[ x \leq x_{\text{obs}} \]
\[ x \geq 0 \]

Figure 14.1. Relationship between econometric and PMP information

An alternative view of the PMP methodology, that is complementary to the “positive” characterization presented above, can be stated as an approach that uses all the available information, whether of the sample and/or expert kind. Hence, in the limit, PMP can be used also in the presence of one single sample observation. We are not advocating the use of one observation if other observations are available. We wish to emphasize only that information should not be discarded only because it is scarce.

This admittedly unorthodox view of a data sample opens up the use of any amount of available sample information and establishes a continuum in the analysis of data samples. In contrast, the traditional econometric methodology requires the availability of a sufficiently large number of degrees of freedom before any estimation is carried out. What should be done, then, when these degrees of freedom are not available? Wait until more observations will somehow become available? In situations such as environmental analyses, development economics, resource economics, it is often difficult to find readily available information in the form of substantial samples. Thus, it is necessary to resort to surveys that will produce only a limited amount of information. In these situations, PMP represents a methodology that should not be overlooked.
Econometric analysis is articulated in two phases: estimation and testing being the first phase, and out of sample prediction or policy analysis the second phase. PMP follows a similar path although it requires three phases: estimation of the output marginal cost, estimation of the cost function or calibration of the nonlinear model, and policy analysis.

To justify the use of a nonlinear model in the calibration phase of PMP let us consider first the dual specification of the LP model (14.3):

\[
\begin{align*}
\text{minimize} \quad TC &= \mathbf{b}'y + \lambda'x_{obs} \\
\text{subject to} \quad A'y + \lambda &\geq p - v \\
y &\geq 0, \lambda \geq 0.
\end{align*}
\]

The meaning of \(\mathbf{b}'y\) is the usual total cost associated with the limiting resources while \(\lambda'x_{obs}\) is the additional total variable cost, above the total accounting cost \(v'x_{obs}\), that varies indeed with the output level. Rearranging the dual constraint

\[A'y + (\lambda + v) \geq p\]  

the marginal cost of the limiting resources, \(A'y\), will be added to the variable marginal cost of the output level, \((\lambda + v)\). Hence, for achieving economic equilibrium, total marginal cost must be greater than or equal to marginal revenue, \(p\).

At this point, it is convenient to recall that a typical dual pair of Quadratic Programming models takes on the following structure:

\[
\begin{align*}
\text{Primal} & \quad \max TNR = p'x - x'Qx/2 \\
\text{Dual} & \quad \min TC = \mathbf{b}'y + x'Qx/2 \\
\text{subject to} \quad Ax &\leq b \\
A'y + Qx &\geq p.
\end{align*}
\]

Comparing the dual constraint in (14.6) with the dual constraint in (14.5) we conclude that, if \((\lambda + v) = Qx\), the dual pair of LP problems (14.3) and (14.4) is equivalent (in the sense that it achieves the same solution) to the dual pair of QP problems (14.6).

To implement this conclusion in the PMP approach we simply restate the equivalence as

\[\lambda_{LP} + v = Qx_{obs}\]  

where the only unknown quantity is the matrix \(Q\). The main justification for postulating relation (14.7) is to allow the variable marginal cost to vary with output. The dimensions of the symmetric matrix \(Q\) are the same as
the number of outputs. Notice that the total cost function is the integral of the marginal cost function and thus

\[ C(x) = \int_0^{x_{obs}} (Qx)'dx = x_{obs}'Qx_{obs}/2 \]

that corresponds to the structure of the cost function in the QP model (14.6).

The estimation of the coefficients of the \( Q \) matrix will be the subject of an extended discussion involving several methodologies, including maximum entropy, as discussed in the next sections. For the moment, and given that we are dealing with only one firm, the simplest approach is to assume that the matrix \( Q \) is diagonal. Hence, the \( j \)-th diagonal coefficient of the \( Q \) matrix, \( Q_{jj} \), can be estimated as

\[ \hat{Q}_{jj} = \frac{\lambda_{LP} + v_j}{x_{obs,j}}. \]  

(14.8)

The purpose of transferring the information from the marginal cost level (\( \lambda_{LP} + v \)) estimated in the linear programming model (14.3) to the matrix \( Q \) of the total cost function \( x'Qx/2 \) is to eliminate the calibrating constraint \( x \leq x_{obs} \) and, therefore, making the resulting nonlinear model more flexible with respect to the parametric analysis that follows. It could be argued that this parametric analysis is akin to the evaluation of the out-of-sample prediction in a traditional econometric analysis. In fact, out-of-sample prediction requires choosing the value of explanatory variables outside the sample and measuring the response on the dependent variable. In a similar vein, parametric programming (or policy analysis) requires choosing the desired values of the model’s parameters (usually either \( p \) or \( b \), or both) and measuring the response on the model’s new solution.

Therefore, the calibrating model of the PMP methodology is stated as

\[
\begin{align*}
\text{maximize} & \quad TNR = p'x - x'\hat{Q}x/2 \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0.
\end{align*}
\]

(14.9)

Given the equivalence stated by equation (14.7), the solution of the QP problem (14.9) is identical to the solution of the LP problem (14.3), that is \( x_{LP} = x_{QP} \) and \( y_{LP} = y_{QP} \). This is the result and the meaning of the calibration process.

After the calibration phase there comes the policy analysis phase. A parametric change in either output prices or resource availability will produce a response of interest to the policy analyst.

To summarize, the PMP methodology was developed for a policy analysis that utilizes all the available information, no matter how scarce. It is
especially suitable in agricultural economics where there is the possibility of combining agronomic and economic data using experts’ information. It is often easier to collect information about the output levels produced on a farm (or by the agricultural sector) than information about the cost of production. Surely, the entrepreneur who decided to produce those output levels must have taken into account the available technology, the market environment, the risky nature of the enterprise. The observed levels of output, therefore, are the result of a complex decision based on a cost function that is known to (or perceived by) the entrepreneur but that is difficult to observe directly.

In general, therefore, the PMP approach consists first of estimating the marginal costs of producing the observed levels of outputs. Secondly, comes the estimation of a parametrically specified cost function. This phase is analogous to the specification of an econometric model. The task here is to select an appropriate functional form for the cost function, an appropriate specification of the distributional assumptions of the various stochastic elements, and an appropriate estimator. Thirdly, the calibrated nonlinear model is used for assessing the response due to parameter variations (prices, resource availability) of interest from a policy viewpoint. Hence, in a stylized fashion:

1. Phase I - Estimation of the output marginal costs,
2. Phase II - Estimation of the cost function,

These are the three phases of the PMP methodology.

PMP with More Than One Observation

In the previous section the PMP methodology was introduced through a discussion of its application to a single firm, or sample observation. This introduction was done mainly with pedagogical reasons in mind. We acknowledge that the event of a single observation is of limited interest as a few observations of a given phenomenon are often available to the researcher. When \( N \) observations (firms) are available, the PMP methodology assumes a slightly more elaborate specification, but it maintains the same objectives and structure.

Phase I - Estimation of Output Marginal Cost

Suppose that the sample at hand represents a group of \( N \) homogeneous firms. By homogeneous firms we intend firms that belong to the same statistical distribution. Hence, those firms need not to be identical, only
reasonably similar. We assume the existence of $J$ outputs in the sample, $j = 1, \ldots, J$. Using the same notation as before for prices, resources and technologies, the phase I model of the $n$-th firm assumes the following structure:

$$\text{max } TNR_n = \mathbf{p}_n' \mathbf{x}_n - \mathbf{v}_n' \mathbf{x}_n$$

subject to

$$A_n \mathbf{x}_n \leq \mathbf{b}_n$$

$$x_{nj} \leq x_{obs, nj} \quad \text{if } x_{obs, nj} > 0$$

$$x_{nj} \leq 0 \quad \text{if } x_{obs, nj} = 0$$

$$\mathbf{x}_n \geq \mathbf{0}$$

where $\mathbf{p}_n$ is the vector of output prices associated with the $n$-th firm, $\mathbf{v}_n$ is the vector of variable accounting costs (for unit of output), $A_n$ is the matrix of technical coefficients, $\mathbf{b}_n$ is the vector of limiting resources, and $\mathbf{x}_{obs,n}$ is the vector of observed levels of output. There are $N$ such LP programs, $n = 1, \ldots, N$. The only unknown quantity is the vector of output levels $\mathbf{x}_n$. This specification admits self selection among firms, that is, not all firms must produce all outputs. The explicit recognition of this realistic event avoids biased estimates of the desired parameters (Heckman). Furthermore, firms are not required to use the same inputs, and the technology is individualized to each firm.

Under the hypothesis of non degeneracy, we partition the vectors of model (14.10) between their produced or “realized” $R$ and “not realized” $NR$ components. Then, the dual problem of the $n$-th firm can be stated as

$$\text{min } TC_n = \mathbf{b}_n' \mathbf{y}_n + \lambda_{Rn}' \mathbf{x}_{Rn}$$

subject to

$$A_{Rn}' \mathbf{y}_n + \lambda_{Rn} + \mathbf{v}_{Rn} = \mathbf{p}_{Rn} \quad \text{for } \mathbf{x}_{Rn} > \mathbf{0}$$

$$A_{NRn}' \mathbf{y}_n + \lambda_{NRn} + \mathbf{v}_{NRn} > \mathbf{p}_{NRn} \quad \text{for } \mathbf{x}_{NRn} = \mathbf{0}$$

where the vectors $\mathbf{y}, \lambda_{Rn}$ and $\lambda_{NRn}$ contain the dual variables of the primal constraints in model (14.10). For the realized (produced) outputs, the dual constraint, representing the corresponding marginal cost and marginal revenue, is satisfied with an equality sign. For the not realized (not produced) outputs (in the non degenerate case), the marginal cost is strictly greater than the marginal revenue, and this is precisely the reason why the entrepreneur has chosen not to produce the corresponding output. This relatively simple LP model, therefore, accounts for the self-selection process that characterizes each unit in a typical sample of firms, no matter how homogeneous. Its dual specification shows explicitly the latent marginal cost of producing or not producing the outputs that are present in the sample as a whole.

The peculiarity of the specification discussed above is that each sample observation (firm) is characterized by its own LP model. This is not
generally true in an econometric model which relies heavily on “average” models. An “average” or, better, an overall sample model is a convenient framework also in the PMP methodology for conducting policy analysis. In particular, by specifying an overall sample model it is possible to define a frontier cost function from which each individual firm’s cost function may deviate by a nonnegative amount.

Hence, let us define the various sample vectors as follows:
- \( p \equiv \text{vector of average output prices,} \)
- \( v \equiv \text{vector of average (variable) accounting costs per unit of output,} \)
- \( A \equiv \text{matrix of average technical coefficients,} \)
- \( b \equiv \text{vector of total sample limiting resources,} \)
- \( x_R \equiv \text{vector of total sample output production.} \)

All the components of the vector \( x_R \) are positive by construction. Hence, the overall model for the sample is not represented by an average firm but, rather, the sample information is considered as one global firm that produces the total sample output using the availability of total resources in the sample. This choice is a matter of researcher’s preference and affects neither the methodology nor the empirical outcome.

Thus, the overall sample LP problem can be stated as

\[
\begin{align*}
\max & \quad TNR = p'x - v'x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \leq x_R \\
& \quad x \geq 0
\end{align*}
\] (14.12)

It should be emphasized that the PMP methodology treats the sample information with maximum care. It does not resort to unnecessary aggregation before estimating any model. On the other hand, it strives to follow the guidelines of econometrics by estimating models for the ultimate purpose of policy analysis.

From the solution of the \((N + 1)\) LP models specified in (14.10) and (14.12) we obtain an estimate of the marginal cost levels associated with the observed output quantities, whether realized or not realized. This completes phase I of the PMP methodology.

**Phase II - Estimation of the Cost Function**

With the information obtained from the LP models of phase I, the marginal cost function for the sample as a whole is stated as

\[
\lambda_{LP} + v = Qx_R
\] (14.13)

where \( Q \) is a symmetric, positive semidefinite matrix. The choice of this particular functional form for the marginal cost function is dictated by simplicity. Numerous other functional forms can be selected as discussed in
a subsequent section. The choice of a functional form at this stage of the PMP methodology is akin to a model specification in econometrics. If the number of sample observations allows it, any traditional test of model specification and selection can be implemented during this phase. The known quantities of relation (14.13) are the vector of marginal costs \((\lambda_{LP} + v)\) and the vector of realized output levels \(x_R\), while the matrix \(Q\) is unknown and must be estimated.

Analogously, the marginal cost function for the \(n\)-th firm is stated as 

\[
\lambda_{LPn} + v_n = Qx_n + u_n \tag{14.14}
\]

where the matrix \(Q\) is the same as in equation (14.13) and \(u_n\) is a nonnegative vector that indexes the marginal cost function of the \(n\)-th firm. In other words, the vector \(u_n\) can be regarded as a deviation from the most efficient marginal cost function corresponding to the overall sample marginal cost function. The nonnegativity of the \(u_n\) vector is established by the frontier character of the overall sample marginal cost function. The components of equation (14.14) are defined as: \(x'_n = (x'_{Rn}, x'_{NRn})\), \(\lambda'_n = (\lambda'_{Rn}, \lambda'_{NRn})\), and \(v'_n = (v'_{Rn}, v'_{NRn})\).

For simplicity, the estimator of choice of this rather demanding econometric problem is a least squares estimator defined as

\[
\min \sum_{n=1}^{N} u'_n u_n / 2 \tag{14.15}
\]

subject to

\[
\begin{align*}
\lambda_{LP} + v &= Qx_R \\
\lambda_{LPn} + v_n &= Qx_n + u_n \\
n &= 1, \ldots, N.
\end{align*}
\]

The matrix \(Q\) is supposed to be a symmetric, positive semi-definite matrix, according to microeconomic theory. In general, without taking special precautions, the estimates of the \(Q\) matrix derived from the solution of the least squares problem (14.15) cannot be expected to satisfy the symmetry and the positive semidefiniteness properties.

In order to achieve this goal, the Cholesky factorization (Benoit, see also Appendix) will be adopted. According to this efficient decomposition, the \(Q\) matrix can be stated as

\[
Q = LHL' \tag{14.16}
\]

where \(L\) is a unit lower triangular matrix and \(H\) is a diagonal matrix with nonnegative elements. It can be shown that the \(Q\) matrix is symmetric, positive semidefinite (definite) if and only if the diagonal elements of \(H\) are nonnegative (positive). These diagonal elements are called Cholesky values (Lau). Therefore, the complete estimator of the matrix \(Q\) (and indexes \(u_n\)) is given by problem (14.15) subject to condition (14.16). A suitable
computer application for estimating this least squares problem with side constraints is GAMS.

The total cost function of the overall sample model can be recovered by integrating the marginal cost function from zero to $x_R$ to yield:

$$C(x) = \int_0^{x_R} (Qx)'dx = x_R'Qx_R/2.$$

Similarly, the cost function of the $n$-th firm will take on the following specification:

$$C_n(x_n) = \int_0^{x_{obs}} (Qx + u_n)'dx_n = x_{obs}'Qx_{obs}/2 + u_n'x_{obs}.$$

By comparing the two cost functions derived above we can conclude that the overall sample cost function represents a frontier cost function while the $n$-th firm cost function deviates from the former by the positive amount $u_n'x_n$. This concludes phase II of the PMP methodology.

**Phase III - Calibrating Model and Policy Analysis**

The least-squares estimates of the matrix $Q$ and vectors $u_n, n = 1, \ldots, N$, which are indicated as $\hat{Q}$ and $\hat{u}_n$, allow for the formulation of a calibrating quadratic programming model that forms the basis for the desired policy analysis. Indeed, there is the possibility of conducting a policy analysis at two levels of response. The first level is at the overall sample, as in a typical econometric study. The second level of policy analysis, better regarded as an extension service analysis, is at the firm level.

Let us begin with the policy analysis at the level of the overall sample. The QP model of interest is stated as

$$\begin{align*}
\max \quad & TNR = p'x - x'\hat{Q}x/2 \\
\text{subject to} \quad & Ax \leq b \\
& x \geq 0.
\end{align*}$$

(14.17)

This model produces primal and dual solutions that are identical to the LP solutions obtained in phase I with model (14.12). This is the essence of calibration. The model can now be used to measure the response on output levels and shadow prices of limiting resources of a variation in either output prices, $p$ or input quantities, $b$. This parametric programming corresponds to the policy analysis envisaged for the PMP methodology from the beginning.
Another calibrating model deals with each sample unit. Because the cost function matrix $Q$ is assumed to be common to each firm in the homogeneous group of sample units, and because the $n$-th $u_n$ vector represents the specific characteristics of the $n$-th firm which force the entrepreneur to deviate from the most efficient cost, the $n$-th firm’s QP calibrating model is stated as

$$\max TNR_n = p'_n x_n - (x'_n Q x_n / 2 + u'_n x_n) \quad (14.18)$$

subject to

$$A_n x_n \leq b_n$$
$$x_n \geq 0.$$

A calibrating model such as (14.18), taken in isolation, is likely to be of limited interest for a policy analysis that involves an entire sector or region. Yet, it is specified in order to reinforce the calibrating power of the PMP methodology and to re-emphasize the fact that PMP treats information without the necessity of making elaborate and aggregate data series before starting the estimation of a model. In principle, the $N$ calibrating models specified by (14.18) could be used in conjunction to the overall model (14.17) for a policy analysis that produces an aggregate response at the sample level and also a consistent disaggregated response at the level of each sample unit.

**Empirical Implementation of PMP**

In the model of phase I, the typical LP specification contains more constraints than decision variables. This is a cause for degeneracy. Assuming $I$ structural constraints, $i = 1, \ldots, I$, it will happen that either some dual variable $y_i$, associated with the structural constraint, $a'_i x \leq b_i$, or some dual variable $\lambda_j$, associated with the calibrating constraints $x_j \leq x_{obs(j)}$, will be equal to zero.

Let us suppose that the vector of resources $b$ represents inputs such as land, labor and capital. In general, it will be of interest to assure that the corresponding dual variables be not equal to zero when the constraints are binding. To accomplish this goal, the option of the researcher is to “transfer” the inevitable degeneracy to the dual variable of some output level. We recall that, if a dual variable such as $\lambda_j$ is equal to zero, it follows that $\lambda_j = p_j - (v_j + a'_j y) = 0$ or, in economic terms, $\lambda_j = MR_j - MC_j(b) = 0$, where we wish to emphasize that the marginal cost $MC_j(b)$ depends on the vector of limiting resources $b$. Since, in a price-taking firm, the marginal revenue is given by the market conditions, the fact that for the specific $j$-th commodity the marginal cost, $MC_j(b)$, is as high as the corresponding marginal revenue means that that commodity is less profitable than other commodities.
In order to accomplish the desired objective (a non-zero dual variable, \( y_i > 0 \), for the structural resources), the phase I LP model will be implemented using an arbitrarily small parameter \( \epsilon \) in the following fashion:

\[
\begin{align*}
\text{maximize} & \quad TNR = p'x - v'x \\
\text{subject to} & \quad Ax \leq b, \quad y \\
& \quad x \leq x_{obs}(1 + \epsilon), \quad \lambda \\
& \quad x \geq 0.
\end{align*}
\] (14.19)

The dimension of the parameter \( \epsilon \) has to be commensurate to the units of measurement of the data series. In general, it can be as small as \( 0.000001 \) but, in order to fulfill the desired goal, its specific dimension must be chosen in relation to the measurement units of the sample information.

**Recovering Revenue and Cost Functions**

The original specification of PMP, as presented in previous sections, was focused on the supply side of the production process. The main objective, in other words, was the recovering of a cost function.

It turns out that, using the same information as specified above, it is also possible to recover a revenue function. This achievement may allow a more articulated and flexible parametric analysis in the final policy phase.

The economic scenario that is envisioned in this case considers the analysis of an entire sector, region, or state where the sector or region is subdivided into many subsectors (or areas). The agricultural sector, with its regional dimension, provides a useful reference point for the scenario discussed here.

Let us suppose, therefore, that we now deal with an extended agricultural region and \( J \) products. We assume that it is possible to subdivide the overall region into \( N \) subregions that can be regarded as extended farms. We suppose that the observed information consists of output prices \( p_n \) and quantities \( \bar{x}_n \) at the farm level, the subsidies per unit of output \( s_n \), the farm unit (accounting) variable costs \( v_n \), the specific farm technology \( A_n \), and the availability of the farm limiting inputs \( b_n \). With this information, it is possible to extend the scope of the PMP methodology to include the estimation of a set of farm level demand functions for the agricultural outputs of the region. To do this, the list of phases of the original PMP methodology must be extended to include a fourth one. We, thus, have

1. Phase I: Estimation of the regional set of farm-level demand functions,
2. Phase II: Estimation of the output marginal costs,
3. Phase III: Estimation of the cost function,
Phase I: Estimation of the Revenue Function

We postulate that the set of farm level demand functions of the $J$ agricultural outputs are specified as the following linear inverse functions

$$p_n = d - D\bar{x}_n + e_n$$

(14.20)

where $d$ is a vector of intercepts, $D$ is a symmetric positive semidefinite matrix, and $e_n$ is a vector of deviations of the $n$-th farm prices from the regional demand functions. The estimation of the demand functions is obtained by a restricted least-squares approach specified a

$$\min \sum_n e'_n e_n / 2$$

(14.21)

subject to

$$p_n = d - D\bar{x}_n + e_n$$

(14.22)

$$D = LGL'$$

(14.23)

$$\sum_n e_n = 0.$$  

(14.24)

Constraint (14.24) guarantees the recovery of the regional demand function $p = d - D\bar{x}$, where $p$ is interpreted as the vector of average prices of the agricultural products prevailing in the region and $\bar{x}$ is the vector of average outputs.

The revenue function of the $n$-th farm is obtained by integrating the estimated demand function as

$$R_n(x_n) = \int_{\bar{x}_n}^{x_n} (\hat{d} - \hat{D}\bar{x}_n + \hat{e}_n)'d\bar{x}_n$$

$$= \hat{d}'\bar{x}_n - \bar{x}_n'\hat{D}\bar{x}_n / 2 + \hat{e}_n'\bar{x}_n.$$  

(14.25)

This process completes phase I of the novel PMP extension.

Phase II: Estimation of Marginal Costs

The estimated revenue function (14.25) is used in phase II to recover the marginal costs of all the $N$ farms, one at a time. This step is similar to model (14.10) but, in this case, the programming problem of the $n$-th farm becomes a quadratic model:

$$\max TNR_n = \hat{d}'x_n - x'_n\hat{D}x_n / 2 + \hat{e}'_nx_n + s'_nx_n - v'_nx_n$$

(14.26)

Dual Variables

subject to

$$A_n x_n \leq b_n$$

Dual Variables

$$y$$

$$x_n \leq \bar{x}_n(1 + \epsilon)$$

$$\lambda$$

$$x_n \geq 0.$$
The entire series of marginal costs of all the \( N \) farms, \( (\hat{\lambda}_n + v_n), n = 1, \ldots, N \), forms the basis for the estimation of the cost function in the next phase.

**Phase III: Estimation of the Cost Function**

The specification of the marginal cost function follows the strategy presented in previous sections, with a variation to admit a vector of intercepts \( \alpha \):

\[
\hat{\lambda}_{QP n} + v_n = \alpha + Q\bar{x}_n + u_n
\]  
\( \text{(14.27)} \)

where \( Q \) is a symmetric, positive semidefinite matrix. The chosen estimator may be a restricted least squares, as in phase I. Hence,

\[
\begin{align*}
\min & \left\{ \sum_n u_n' u_n / 2 \right\} \\
\text{subject to} & \quad \hat{\lambda}_{QP n} + v_n = \alpha + Q\bar{x}_n + u_n \\
& \quad Q = LGL'
\end{align*}
\]  
\( \text{(14.28)} \)

\( \text{(14.29)} \)

\( \text{(14.30)} \)

The cost function of the \( n \)-th farm is obtained by integration of the estimated marginal cost function, as before:

\[
C_n(x_n) = \int_0^{\bar{x}_n} (\hat{\alpha}' + \hat{Q}x_n + \hat{u}_n)' dx_n \\
= \hat{\alpha}' \bar{x}_n + \bar{x}_n' \hat{Q} \bar{x}_n / 2 + \bar{u}_n' \bar{x}_n.
\]  
\( \text{(14.31)} \)

**Phase IV: Calibrating Model**

The final assembly of the calibrating model is done by defining an objective function that is the difference between the revenue and the cost functions estimated above:

\[
\max \ TNR_n = \hat{\alpha}'x_n - x_n' \hat{D} x_n / 2 + \hat{e}_n' x_n + s_n' x_n \\
- (\hat{\alpha}'x_n + x_n' \hat{Q} x_n / 2 + \hat{u}_n' x_n)
\]  
\( \text{(14.32)} \)

subject to

\[
\begin{align*}
A_n x_n & \leq b_n \\
\quad x_n & \geq 0.
\end{align*}
\]

This model can now be used to evaluate policy changes involving, for example, the farm subsidies \( s_n \) and the limiting inputs \( b_n \).
Symmetric Positive Equilibrium Problem - SPEP

In previous sections, the specification of the PMP methodology involved essentially one limiting input, say land. This restriction has to do with the degeneracy caused in recovering the input and outputs marginal costs. With more constraints than decision variables, the estimation of marginal cost phase of previous sections produces a zero value of either the dual variable of the single limiting input, \( y \), or (at least) one zero value in the vector of marginal costs, \( \lambda \). The specification of an appropriate dimension of the \( \epsilon \) parameter of the calibrating constraints allows to obtain a positive value of the dual variable associated with the single structural constraint, as we would like to achieve for a constraint such as land. When multiple structural constraints are present, the degeneracy of some of the dual variables associated with them cannot be avoided.

To avoid this “pitfall” of PMP, it is necessary to state a symmetric structure and formulate the economic problem as an equilibrium model. In order to grasp the meaning of the relevant symmetric structure and the associated equilibrium problem we review the primal and dual constraints of model (14.3) which, now we know, admit only one limiting input \( b \):

\[
\begin{align*}
Ax &\leq b & y & (14.33) \\
x &\leq \bar{x} & \lambda & (14.34) \\
A'y + (\lambda + \nu) &\geq p & x & (14.35) \\
y &\geq 0 & & (14.36)
\end{align*}
\]

The economic meaning of all the above terms remains as before. The vector \( \bar{x} \) stands now for the realized (observed) quantities of the firm’s outputs. The structure of problem (14.33)-(14.36) is asymmetric because the primal constraints exhibit only one set of primal variables, \( x \), while the dual constraints include two set of dual variables, \((y, \lambda)\).

As stated above, the introduction of multiple limiting inputs induces a degeneracy of the associated dual variables. To avoid this occurrence, we assume that information on the regional market price of all limiting inputs is available and observable. Let vector \( r \) represent this information. Then, the Symmetric Positive Equilibrium Problem (SPEP) can be stated as

\[
\begin{align*}
Ax + \beta &\leq b & y & (14.37) \\
x &\leq \bar{x} & \lambda & (14.38) \\
A'y + (\lambda + \nu) &\geq p & x & (14.39) \\
y &\geq r & \beta & (14.40)
\end{align*}
\]

The above four constraints have a symmetric formulation. The first two constraints, (14.37) and (14.38), are quantity constraints representing,
respectively, the demand and supply of limiting inputs and output calibration. Constraints (14.39) and (14.40) represent marginal costs and marginal revenues of outputs and inputs, respectively. Notice the symmetric position and role of the two vector variables designated with Greek symbols, \( \lambda \) and \( \beta \). Both vectors have the meaning of dual variables of the corresponding constraints. The \( \lambda \) vector variable appears also in the output marginal cost constraint (14.39) and, together with vector \( v \), takes on the meaning of variable marginal cost. The \( \beta \) dual vector variable appearing in the input constraint (14.37) defines the effective supply of limiting inputs \( (b - \beta) \), as opposed to the fixed input supply represented by the vector \( b \). The vector \( \beta \) enables a generalization of the limiting input supply that allows for a more flexible specification of allocatable inputs. The vector \( b \) can be thought of as an upper limit on the quantity of allocatable inputs that does not imply a zero implicit marginal cost for quantities of allocatable inputs used in amount less than \( b \), as in the usual asymmetric specification \( Ax \leq b \). Furthermore, the vector \( \beta \) is not to be regarded as a slack variable. The symbols to the right of the four constraints are their corresponding dual variables.

To complete the specification of the equilibrium problem we must state the associated complementary slackness conditions:

\[
y'(b - Ax - \beta) = 0 \quad (14.41)
\]
\[
\lambda'(\bar{x} - x) = 0 \quad (14.42)
\]
\[
x'(A'y + \lambda + v - p) = 0 \quad (14.43)
\]
\[
\beta'(y - r) = 0 \quad (14.44)
\]

The solution of the above Symmetric Positive Equilibrium Problem represented by the set of constraints (14.37)-(14.44) generates estimates of output levels, \( x \), the effective supply of limiting inputs, \( (b - \beta) \), the total marginal cost of production activities, \( (A'y + \lambda + v) \), and the marginal cost of limiting inputs, \( y \).

In order to solve the above SPEP by means of a suitable computer code such as GAMS, it is convenient to introduce slack vector variables in each constraint (14.37)-(14.40). Let \( z_{P1}, z_{P2}, z_{D1}, z_{D2} \) be nonnegative slack vectors of the primal and dual constraints of the SPEP structure. Then, a computable specification of SPEP can be stated as

\[
\min \{ z_{P1}'y + z_{P2}'\lambda + z_{D1}'x + z_{D2}'\beta \} = 0 \quad (14.45)
\]

subject to

\[
Ax + \beta + z_{P1} = b \quad (14.46)
\]
\[
x + z_{P2} = \bar{x} \quad (14.47)
\]
\[
A'y + (\lambda + v) = p + z_{D1} \quad (14.48)
\]
\[
y = r + z_{D2}. \quad (14.49)
\]
In this specification of SPEP, the complementary slackness conditions (14.41)-(14.44) have been added together to form the auxiliary objective function (14.45) that, we know, must obtain a zero value at the equilibrium solution since all the variables are nonnegative. Hence, an equilibrium solution \((x, y, \lambda, \beta)\) is achieved when the objective function (14.45) reaches a value of zero and the solution vector \((x, y, \lambda, \beta)\) is feasible.

**Phase II of SPEP: the Total Cost Function**

In previous versions of the traditional PMP methodology, phase II (or phase III) of the estimation process dealt with the recovery of a variable cost function, that is a function concerned with variable costs. SPEP, however, will recover a total cost function. Let us recall that total marginal cost is represented by \([A'y + (\lambda + v)]\), with component \((\lambda + v)\) representing the variable marginal cost while \(A'y\) has the meaning of marginal cost due to limiting inputs.

By definition, a total cost function is a function of output levels and input prices, \(C(x, y)\). The properties of a cost function require it to be concave and linear homogeneous in input prices. The functional form selected to represent input prices is a generalized Leontief specification with nonnegative and symmetric off-diagonal terms. For the outputs, the functional form is a quadratic specification in order to avoid the imposition of a linear technology. Furthermore, we must allow sufficient flexibility to fit the available empirical data. For this reason we add an unrestricted intercept term, with the proviso to guarantee the linear homogeneity of the entire cost function. All these considerations lead to the following functional form:

\[
C(x, y) = u'y'(f'x) + u'y(x'Qx)/2 + y^{1/2}Sy^{1/2} \tag{14.50}
\]

where \(u\) is a vector of unit elements. Many different functional forms could have been selected. The matrix \(Q\) is symmetric positive semidefinite while the \(S\) matrix is symmetric with nonnegative off-diagonal terms.

The marginal cost function is the derivative of equation (14.50) with respect to the output vector \(x\), that is

\[
\frac{\partial C}{\partial x} = (u'y)f' + (u'y)Qx = A'y + \lambda + v \tag{14.51}
\]

and, by Shephard lemma, the limiting input derived demand functions are

\[
\frac{\partial C}{\partial y} = (f'x)u + u(x'Qx)/2 + \Delta_y^{-1/2}Sy^{1/2} = Ax = b - \beta. \tag{14.52}
\]

The matrix \(\Delta_y^{-1/2}Sy^{1/2}\) is diagonal with elements of the vector \(y^{-1/2}\) on the diagonal.
The objective of phase II is to estimate the parameters of the cost function $f, Q, S$. This estimation can be performed by a maximum entropy estimator (Golan et al., 1996). The relevant relations are stated as follows

$$A'y + \hat{\lambda} + v = (u'y)f + (u'y)Q\bar{x}$$  
$$Ax = (f'\bar{x})u + u(\bar{x}'Q\bar{x})/2 + \Delta_{y-1/2}S\bar{y}^{1/2}$$  

Equation (14.53) represents output marginal costs. Equation (14.54) is the vector of derived demands for inputs. The sample information is given by the left-hand-side expressions of equations (14.53) and (14.54), where $\bar{x}$ is the realized level of outputs and ($\hat{y}, \hat{\lambda}$) are the computed shadow variables from phase I.

**Phase III of SPEP: Calibrating Model for Policy Analysis**

With the estimated parameters of the total cost function, $\hat{f}, \hat{Q}, \hat{S}$, it is possible to set up a calibrating specification that takes the structure of an equilibrium problem:

$$\min \{z'_{P1}y + z'_{D1}x + z'_{D2}\beta\} = 0$$  
subject to

$$(\hat{f}'x)u + u(x'\hat{Q}x)/2 + \Delta_{y-1/2}\hat{S}y^{1/2} + \beta + z_{P1} = b$$  
$$(u'y)f + (u'y)Qx = p + z_{D1}$$  
$$y = r + z_{D2}.$$  

where all the unknown variables are nonnegative. The linear technology in equation (14.46) is now replaced by the Shepard lemma’s equation (14.52) in constraint (14.56). Similarly, the marginal cost in equation (14.48) is replaced by the marginal output cost equation (14.51). In addition, the calibration constraints in equation (14.47) are removed since the calibration of output levels is now guaranteed by the cost function. Constraint (14.56), when rearranged as $\{(\hat{f}'x)u + u(x'\hat{Q}x)/2 + \Delta_{y-1/2}\hat{S}y^{1/2} \leq b - \beta\}$, states the quantity equilibrium condition according to which the demand for limiting inputs must be less than or equal to the effective supply of those inputs. The quantity $(b - \beta)$ is the effective supply by virtue of the endogenous parameter $\beta$ that is part of the solution. The SPEP specification given above neither implies nor excludes an optimizing behavior. Hence, it removes from the original PMP methodology the last vestige of a normative behavior. Furthermore, the nonlinear cost function implies a nonlinear technology and, therefore, it removes the restriction of fixed production coefficients.
Dynamic Positive Equilibrium Problem - DPEP

The methodology of Symmetric Positive Equilibrium Problem is extended in this section to include a dynamic structure. Dynamic models of economic problems can take on different specifications in relation to different sources of dynamic information. When dealing with farms whose principal output is derived from orchards, for example, the equation of motion is naturally represented by the difference between standing orchard acreage in two successive years plus new planting and minus culling. In more general terms, the investment model provides a natural representation of stocks and flows via the familiar investment equation

\[ K_t = K_{t-1} + I_t - \delta K_{t-1} \]  

where \( K_t \) represents the capital stock at time \( t \), \( I_t \) is the investment flow at time \( t \), and \( \delta \) is the fixed depreciation rate. This dynamic framework, expressed by an equation of motion, becomes operational only when explicit information about investment, initial stock and depreciation is available.

Annual crops are also dynamically connected through decisions that involve price expectations and some inertia of the decision making process. We observe that farmers who produce field crops, for example, will produce these activities year after year with an appropriate adjustment of both acreage and yields. In this section, therefore, we consider economic units (farms, regions, sectors) that produce annual crops. That is, production activities which, in principle, may have neither technological antecedents nor market consequences but, nevertheless, are observed to be connected through time. We postulate that the underlying dynamic connection is guided by a process of output price expectations.

The Dynamic Framework

We assume that output price expectations of the decision maker are governed by an adaptive process such as:

\[ p_t^* - p_{t-1}^* = \Gamma(p_t - p_{t-1}) \]

where the starred vectors are interpreted as expected output prices and \( \Gamma \) is a diagonal matrix with unrestricted elements. In general, the elements of the \( \Gamma \) matrix are required to be positive and less than 1 in order to guarantee the stability of the difference equation in an infinite series of time periods. The case discussed here, however, considers a finite horizon of only a few years and no stability issue is at stake. It is as if we were to model an arbitrarily small time interval of an infinite horizon. Within
such a small interval, the relation expressed by equation (14.60) can be either convergent or explosive without negating the stability of the infinite process. A further assumption is that the expected output supply function is specified as

\[ x_t = B p_t^* + w_t \]  

(14.61)

where \( B \) is a positive diagonal matrix and \( w_t \) is a vector of intercepts. Then, equation (14.60) can be rearranged as

\[ \Gamma p_{t-1} = p_t^* - [I - \Gamma] p_{t-1}^* \]  

(14.62)

while, by lagging one period the supply function, multiplying it by the matrix \([I - \Gamma]\), and subtracting the result from equation (14.61), we obtain

\[ x_t - [I - \Gamma] x_{t-1} = B\{p_t^* - [I - \Gamma] p_{t-1}^*\} + w_t - [I - \Gamma] w_{t-1} \]  

(14.63)

where \( v_t = w_t - [I - \Gamma] w_{t-1} \). Hence, the equation of motion involving annual crops and resulting from the assumption of adaptive expectations for output prices is

\[ x_t = [I - \Gamma] x_{t-1} + B \Gamma p_{t-1} + v_t. \]  

(14.64)

It is important to emphasize that this equation of motion is different from the more traditional dynamic relation where the state variable is usually interpreted as a stock and the control is under the jurisdiction of the decision maker. Equation of motion (14.64) emerges from an assumption of adaptive expectations about output prices. Prices are not under the control of the decision maker and, furthermore, the state variable is not a stock but a flow variable as it represents yearly output levels. Nevertheless, relation (14.64) is a legitimate equation of motion that relates entrepreneur’s decisions from year to year.

**Phase I of DPEP: Estimation of Marginal Costs**

Before proceeding further in the development of the Dynamic Positive Equilibrium Problem it is necessary to estimate matrices \( B \) and \( \Gamma \) and vector \( v_t \) that define the equation of motion. One approach to this problem is to use the maximum entropy methodology as presented by Golan et al. (1996). In this section, therefore, we assume that the estimation of the relevant matrices was performed, \( (\hat{B}, \hat{\Gamma}, \hat{v}_t) \), and proceed to the discussion of phase I of DPEP.
We begin with a specification of the optimization problem for the entire horizon $t = 1, \ldots, T$ and the statement of a salvage function. We assume that the economic agent wishes to maximize the discounted stream of profit (or net revenue) over the horizon $T$. After $T$ periods, it is assumed that the objective function consists of the discounted value of profit from period $(T+1)$ to infinity, which is realized under a condition of steady state. Furthermore, the given information refers to input and output prices, $r_t, p_t$, the discount rate, $\rho$, the matrix of technical coefficients, $A_t$, and the available supply of limiting inputs, $b_t$.

Analytically, then, the discrete Dynamic Positive Equilibrium Problem takes on the following specification

$$
\max V = \sum_{t=1}^{T} \left[ p_t' x_t - r_t' (b_t - \beta_t) \right] / (1 + \rho)^{t-1} \\
+ \sum_{\tau=T+1}^{+\infty} \left[ p_{T+1}' x_{T+1} - r_{T+1}' (A_{T+1} x_{T+1}) \right] \left[ \frac{1}{1 + \rho} \right]^{\tau-1}
$$

subject to

$$
A_t x_t + \beta \leq b_t, \quad t = 1, \ldots, T \\
x_t = [I - \hat{\Gamma}] x_{t-1} + \hat{B} \hat{\Gamma} p_{t-1} + \hat{\psi}_t, \quad t = 1, \ldots, T + 1
$$

Constraint (14.66) expresses the technological requirements for producing the vector of crop activities $x_t$ given the limiting resource availability $b_t$. The objective function is stated in two parts. The first component expresses the discounted profit over the entire horizon $T$. The second component is the salvage function. By defining $d^T = \left[ \frac{1}{1 + \rho} \right]^T$, under the assumption of steady state, the salvage function may be written as

$$
\sum_{\tau=T+1}^{+\infty} \left[ p_{T+1}' x_{T+1} - r_{T+1}' (A_{T+1} x_{T+1}) \right] \left[ \frac{1}{1 + \rho} \right]^{\tau-1} = 0
$$

As stated above, we assume that in period $(T+1)$, the period after planning ceases, the variables are at their steady state values.
With these stipulations, the Lagrangean function of problem (14.65)-(14.67) is stated as

\[
L = \sum_{t=1}^{T} \left\{ p'_t x_t - r'_t (b_t - \beta_t) \right\} / (1 + \rho)^{(t-1)}
+ \left[ p'_{T+1} x_{T+1} - r'_{T+1} (A_{T+1} x_{T+1}) \right] / \rho (1 + \rho)^{(T-1)}
+ \sum_{t=1}^{T} (b_t - \beta_t - A_t x_t)' y_t
+ \sum_{t=1}^{T+1} \left\{ [I - \hat{\Gamma}] x_{t-1} + \hat{B} \hat{\Gamma} p_{t-1} + \hat{v}_t - x_t \right\}' \lambda_t.
\]

(14.68)

The corresponding KKT conditions are

\[
\frac{\partial L}{\partial x_t} = p_t / (1 + \rho)^{t-1} - A'_t y_t - \lambda_t + [I - \hat{\Gamma}] \lambda_{t+1} \leq 0
\] (14.69)

\[
\frac{\partial L}{\partial x_{T+1}} = (p_{T+1} - A'_{T+1} r_{T+1}) / \rho (1 + \rho)^{(T-1)} - \lambda_{T+1} = 0
\] (14.70)

\[
\frac{\partial L}{\partial \beta_t} = r_t / (1 + \rho)^{t-1} - y_t \leq 0
\] (14.71)

\[
\frac{\partial L}{\partial \lambda_t} = [I - \hat{\Gamma}] x_{t-1} + \hat{B} \hat{\Gamma} p_{t-1} + \hat{v}_t - x_t = 0
\] (14.72)

\[
\frac{\partial L}{\partial y_t} = b_t - \beta_t - A_t x_t \geq 0
\] (14.73)

together with the associated complementary slackness conditions.

This discrete dynamic problem can be solved, year by year, using a backward solution approach on the system of KKT conditions (14.69)-(14.73). The key to this strategy is the realization that the equation of motion calibrates exactly the sample information, \(\hat{x}_t\), for any year, that is, \(\hat{x}_t = [I - \hat{\Gamma}] x_{t-1} + \hat{B} \hat{\Gamma} p_{t-1} + \hat{v}_t\) and, therefore, the left-hand-side quantity \(\hat{x}_t\) can replace the corresponding right-hand-side expression. In other words, we can equivalently use the available and contemporaneous information about the economic agent’s decisions. Furthermore, from KKT condition (14.70), the costate variable \(\lambda_{T+1}\), for the time period outside the horizon, is equal to the derivative of the salvage function with respect to the decisions at time \((T + 1)\), \(\dot{\lambda}_{T+1} = (p_{T+1} - A'_{T+1} r_{T+1}) / \rho (1 + \rho)^{(T-1)}\).
Then, at time $T$, the equilibrium problem to be solved is given by the following structural relations

\begin{align}
A_T x_T + \beta_T & \leq b_T & y_T \geq 0 \quad (14.74) \\
x_T \leq \bar{x}_T & = [I - \hat{\Gamma}] x_{T-1} + \tilde{B} \hat{\Gamma} p_{T-1} + \psi_T & \lambda_T \geq 0 \quad (14.75) \\
A'_T y_T + \lambda_T & \geq p_T d(T-1)/\rho + [I - \hat{\Gamma}] \hat{\lambda}_{T+1} & x_T \geq 0 \quad (14.76) \\
y_T & \geq r_T d(T-1) & \beta_T \geq 0 \quad (14.77)
\end{align}

and by the associated complementary slackness conditions. Knowledge of the realized levels of output at time $t$, $\bar{x}_t$, and of the costate variables $\hat{\lambda}_{t+1}$, estimated at time $t + 1$, allows for the solution of the dynamic problem as a sequence of $T$ equilibrium problems. Hence, the dynamic linkage between successive time periods is established through the vector of costate variables $\hat{\lambda}_t$. In this way, the equilibrium problem (14.74)-(14.77) can be solved backward to time ($t = 1$) without the need to specify initial conditions for the vector of state variables $x_0$ and the price vector $p_0$. This dynamic problem arises exclusively from the assumption of adaptive price expectations and, given the DPEP as stated above, the costate variable $\hat{\lambda}_t$ does not depend explicitly upon the state variable $x_t$. This implies that the positive character of the problem, with the concomitant use of the realized levels of activity outputs $\bar{x}_t$, avoids the usual two-part solution of a dynamic problem where the backward solution is carried out in order to find the sequence of costate variables $\hat{\lambda}_t$, and the forward solution is devoted to finding the optimal level of the state variables $x_t$. In the context specified here, the solution regarding the state variable $x_t$ is obtained contemporaneously with the solution of the costate variable $\hat{\lambda}_t$.

The objective of DPEP during phase I, therefore, is to solve $T$ equilibrium problems starting at the end point of the time horizon $t = T, T - 1, \ldots, 2, 1$ and having the following structure:

\begin{align}
\min \{ z'_T y_T + z'_P \lambda_t + z'_D \bar{x}_t + z'_D \beta_t \} = 0 \quad (14.78)
\end{align}

subject to

\begin{align}
A_t x_t + \beta_t + z'_P \lambda_t & = b_t \quad (14.79) \\
x_t + z'_P \bar{x}_t & = \bar{x}_t \quad (14.80) \\
A'_t y_t + \lambda_t & = p_t d(t-1) + [I - \hat{\Gamma}] \hat{\lambda}_{t+1} + z'_D \bar{x}_t \quad (14.81) \\
y_t & = r_t d(t-1) + z'_D \beta_t \quad (14.82)
\end{align}

The objective function is the sum of all the complementary slackness conditions. A solution of the equilibrium problem is achieved when the objective function reaches the zero value. The main objective of phase I is the recovery of the costate variables for the entire horizon and of the dual variables for the structural primal constraints to serve as information in the estimation of the cost function during the next phase. The fundamental reason...
for estimating a cost function to represent the economic agent’s decision process is to relax the fixed-coefficient technology represented by the $A_t$ matrix and to introduce the possibility of a more direct substitution between products and limiting inputs. In other words, the observation of output and input decisions at time $t$ provides only a single point in the technology and cost spaces. The process of eliciting an estimate of the latent marginal cost levels and the subsequent recovery of a consistent cost function which rationalizes the available data is akin to the process of fitting an isocost through the observed output and input decisions.

**Phase II of DPEP: Estimation of the Cost Function**

By definition, total cost is a function of output levels and input prices. In a dynamic problem, the total cost function is defined period by period as in a static problem and represented as $C(x_t, y_t, t) = C_t(x_t, y_t)$. The properties of the cost function of a dynamic problem follow the same properties specified for a static case (Stefani, 1989): it must be concave and linearly homogeneous in input prices in each time period. Borrowing, therefore, from the SPEP specification discussed in previous sections, the functional form of the cost function for DPEP is stated as

$$C_t(x_t, y_t) = u'y_t(f_t(x_t) + u'y_t(x_t'Q_t x_t)/2 + y_t^{1/2} S_t^{1/2} y_t^{1/2})$$

where $u$ is a vector of unitary elements. The matrix $Q$ is symmetric positive semidefinite while the $S$ matrix is symmetric with nonnegative elements on and off the main diagonal.

The marginal cost function at time $t$ is the derivative of equation (14.83) with respect to the output level at time $t$, that is

$$\frac{\partial C_t}{\partial x_t} = (u'y_t f_t + u'y_t x_t'Q_t x_t) = A_t y_t$$

while, by Shephard lemma, the limiting input derived demand functions are

$$\frac{\partial C_t}{\partial y_t} = (f_t'x_t)u + u(x_t'Q_t x_t)/2 + \Delta_{y_t}^{-1/2} S_t^{1/2} y_t^{1/2} = A_t x_t.$$

The matrix $\Delta_{y_t}^{-1/2} S y_t^{1/2}$ is diagonal with elements of the vector $y_t^{-1/2}$ on the diagonal.

There is a significant difference between the marginal cost of the static equilibrium problem and the short-run (period by period) marginal cost of the dynamic equilibrium problem. If one considers the static equilibrium specification formulated in model (14.37)-(14.40), the marginal cost
is given in relation (14.39) as $MC(x, y) \equiv A' y + (\lambda + v)$. In other words, without a time dimension, the marginal cost is equal to the sum of the marginal cost attributable to the limiting inputs, $A' y$, plus the variable marginal cost attributable to the output levels, $(\lambda + v)$. In the dynamic context specified above, the Lagrange multiplier $\lambda_t$ assumes the meaning of costate variable and signifies the marginal valuation of the state variable $x_t$. Its forward looking nature is expressed by the following relation $\lambda_{T-n} = \sum_{s=0}^{n+1} (I-\tilde{\Gamma})^s (p_{T-n+s} - A'_{T-n+s} y_{T-n+s})$, where $n = -1, 0, 1, \ldots, T$.

In a dynamic context, therefore, the costate variable $\lambda_t$ cannot be used to define the period-by-period marginal cost (as done in a static equilibrium problem where the symbol $\lambda$ is interpreted simply as variable marginal cost) because it incorporates the long-run notion of a trajectory associated with a multi-period horizon. In the dynamic equilibrium problem discussed above, the period-by-period marginal cost is defined as $MC(x_t, y_t) \equiv A'_t y_t$, as indicated in relation (14.84).

The objective of phase II is to estimate the parameters of the cost function given in equation (14.83), $f_t, Q_t, S_t$. This estimation can be performed with a maximum entropy approach (Golan et al., 1996).

**Phase III of DPEP: Calibration and Policy Analysis**

The estimated cost function can now be used to replace the marginal cost levels and the demand for inputs in the dynamic equilibrium problem of phase I. This replacement assures the calibration of the model, relaxes the fixed-coefficient technology, and allows direct substitution among inputs and outputs. At this stage, therefore, it is possible to implement policy scenarios based upon the variation of output and input prices.

The structure of the calibrating DPEP is given below. With knowledge of the costate variables $\hat{\lambda}_t$ and $\hat{\lambda}_{t+1}$ obtained from the solution of the DPEP of phase I, the following specification calibrates the output decisions and the input dual variables for any period:

\[
\min \{z'_{P1,t} y_t + z'_{D1,t} x_t + z'_{D2,t} \beta_t\} = 0 \tag{14.86}
\]

subject to

\[
(u' y_t) \hat{f}_t + (u' y_t) \hat{Q}_t x_t = p_t d^{-1} + [I - \tilde{\Gamma}] \hat{\lambda}_{t+1} - \hat{\lambda}_t + z_{D1,t} \tag{14.87}
\]

\[
y_t = r_t d^{-1} + z_{D2,t}. \tag{14.88}
\]

The use of the costate values obtained during phase I is required for eliminating the constraint on the decision variables, $x_t \leq x_{Rt}$, which were employed in phase I specification precisely for eliciting the corresponding values of the costates. As observed above, the costate variables are the
dynamic link between any two periods and their determination requires a backward solution approach. If we were to require their measurement for a second time during the calibrating and policy analysis phase, we would need to add also the equation of motion in its explicit form, since the constraint \( x_t \leq \bar{x}_t \) would no longer be sufficient. In this case, the \( T \) problems would be all linked together and ought to be solved as a single large-scale model. The calibration phase, therefore, is conditional upon the knowledge of the costate variables obtained during phase I and involves the period-by-period determination of the output decisions and dual variables of the limiting inputs.

Given the dynamic structure of the model, a policy scenario becomes a prediction at the end of the \( T \)-period horizon. All the model components at period \( T \) are known and the researcher wishes to obtain a solution of the DPEP for the \((T + 1)\) period. The parameters of the cost function are assumed constant and equal to those at time \( T \). The costate variables at times \((T + 2)\) and \((T + 1)\), \( \hat{\lambda}_{T+2}, \hat{\lambda}_{T+1} \), are taken to be equal to the steady state marginal values of the salvage function. The remaining parameters, \( b_{T+1}, r_{T+1} \) and \( p_{T+1} \) will assume the value of interest under the desired policy scenario. More explicitly, the relevant structure of the dynamic positive equilibrium problem during the policy analysis phase takes on the following specification:

\[
\begin{align*}
\min \{ & z'_{P1,T+1}y_{T+1} + z'_{D1,T+1}x_{T+1} + z'_{D2,T+1}\beta_{T+1} \} = 0 \quad \text{ (14.90)} \\
\text{subject to} \quad & (\hat{f}'_{T+1}x_{T+1})u + u(x'_{T+1}\hat{Q}_{T+1}x_{T+1})/2 + \Delta y_{T+1}^{1/2}S_{T+1}y_{T+1}^{1/2} \\
& + \beta_{T+1} + z_{P1,T+1} = b_{T+1} \quad \text{ (14.91)} \\
& (u'y_{T+1})\hat{f}_{T+1} + (u'y_{T+1})\hat{Q}_{T+1}x_{T+1} = p_{T+1}d' \\
& + [I - \hat{\Gamma}]\hat{\lambda}_{T+2} - \hat{\lambda}_{T+1} + z_{D1,T+1} \quad \text{ (14.92)} \\
y_{T+1} = r_{T+1}d' + z_{D2,T+1}. \quad \text{ (14.93)}
\end{align*}
\]

Projected policy prices, either on the input or output side, will induce responses in output and input decisions that are consistent with a process of output price expectations articulated in previous sections.
Appendix - Cholesky Factorization

**Definition:** Given a symmetric, positive semidefinite matrix $Q$, its Cholesky factorization (decomposition) is stated as follows:

\[ Q = LHL' \]  

where $L$ is a unit lower triangular matrix and $H$ is a diagonal matrix with nonnegative elements. The elements of the $H$ matrix are called Cholesky values.

**Example.** Consider a $(3 \times 3)$ SPSD (symmetric, positive semidefinite) matrix $Q$. Its Cholesky factorization is stated as

\[
\begin{bmatrix}
q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
l_{21} & 1 & 0 \\
l_{31} & l_{32} & 1
\end{bmatrix} \begin{bmatrix}
h_{11} & 0 & 0 \\
h_{22} & 0 & 0 \\
0 & 0 & h_{33}
\end{bmatrix} \begin{bmatrix}
1 & l_{21} & l_{31} \\
0 & 1 & l_{32} \\
0 & 0 & 1
\end{bmatrix}
\]

It follows that:

\[
\begin{align*}
q_{11} &= h_{11} \\
q_{12} &= h_{11}l_{12} \\
q_{13} &= h_{11}l_{13} \\
q_{22} &= h_{22} + h_{11}(l_{21})^2 \\
q_{23} &= h_{22}l_{32} + h_{11}l_{21}l_{31} \\
q_{33} &= h_{33} + h_{11}(l_{31})^2 + h_{22}(l_{32})^2
\end{align*}
\]

From the example, it is clear that, by knowing the coefficients of the $Q$ matrix, it is possible to derive, one at a time, all the coefficients of the $H$ and $L$ matrices.

In the PMP methodology, however, the Cholesky factorization is used in a reverse mode. The $Q$ matrix is not known and must be estimated in such a way that it should turn out to be symmetric and positive semidefinite. This is dictated by economic theory. Hence, a feasible approach to that task is to estimate the coefficients of the $L$ and $H$ matrices and make sure that the diagonal elements of the $H$ matrix are nonnegative. This can easily be done with a computer program like GAMS.
Bibliographical References


