13. Two-Person Zero- and Non-Zero-Sum Games

Game theory deals with conflict of interest between (among) persons. A game is a situation of conflict in which two or more persons interact by choosing an admissible set of actions while knowing the reward associated with each action. The persons who interact are called *players*, the set of actions are called *strategies* and the rewards are called *payoffs*. Hence, a game is a set of rules describing all the possible actions available to each player in correspondence with the associated payoff. In a game, it is assumed that each player will attempt to optimize his/her (expected) payoff.

In this chapter, we will discuss two categories of games that involve two players, player 1 and player 2. The first category includes zero-sum games because the total payoff awarded the two players is equal to zero. In other words, the "gain" of one player is equal to the "loss" of the other player. This type of games assumes the structure of a dual pair of linear programming problems. The second category includes games for which the total payoff is not equal to zero and each player may have a positive payoff. This type of games requires the structure of a linear complementarity problem in what is called a bimatrix game.

The notion of strategy is fondamental in game theory. A strategy is the specification of all possible actions that a player can take for each move of the other player. In general, a player has available a large number of strategies. In this chapter we will assume that this number is finite. Player 1 and player 2 may have a different number of strategies. A set of strategies describes all the alternative ways to play a game.

There are *pure* strategies and *mixed* strategies. Pure strategies occur when chance does not influence the outcome of the game, that is, the outcome of the game is entirely determined by the choices of the two players. In the case of a game with pure strategies, a_{ij} indicates the payoff of the game when player 1 chooses pure strategy i and player 2 chooses pure strategy

j. Mixed strategies occur when chance influences the outcome of a game. In this case it is necessary to talk about an *expected* outcome in the form of an *expected* payoff.

Two-Person Zero-Sum Games

In a two-person zero-sum game, it is possible to arrange the payoffs corresponding to all the available finite strategies in a matrix A exhibiting m rows and n columns

$$\begin{array}{c} \\ \text{Player } 1 \Rightarrow i \end{array} \overset{\text{Player } 2 \Rightarrow j}{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}} = A$$

We will say that player 1, or row player, or P1, has m strategies available to him, while player 2, or column player, or P2, has n strategies available to her. It is assumed that player 1 attempts to win as much as possible, that is, he wishes to maximize his payoff while player 2 will attempt to minimize the payoff of player 1. This assumption corresponds to the *MaxiMin-MiniMax* principle. This principle may be described as follows. If P1 will choose the i strategy, he will be assured of winning at least

$$\min_j a_{ij} \tag{13.1}$$

regardless of the choice of P2. Therefore, the goal of player 1 will be to choose a strategy that will maximize this amount, that is,

$$\max_i \min_j a_{ij}. \tag{13.2}$$

Player 2 will act to limit the payoff of player 1. By choosing strategy j, P2 will be assured that P1 will not gain more than

$$\max_i a_{ij} \tag{13.3}$$

regardless of the choice of P1. Therefore - given that what P1's gain corresponds to P2's loss, according with the stipulation of the game - P2 will choose a strategy to minimize her own loss, that is,

$$\min_j \max_i a_{ij}. \tag{13.4}$$

The following discussion borrows from Headly. There are several ways to restate the behavior of the two players. The expression in (13.1) is a lower bound, or a floor, on the payoff of player 1. Hence, expression (13.2) can be interpreted as a maximization of the lower bound payoff. Analogously, expression (13.3) may be interpreted as an upper bound, or ceiling, on the amount lost by player 2. Hence, expression (13.4) may be regarded as the minimization of that ceiling.

If there exists a payoff amount, say, a_{rk} that will correspond to

$$a_{rk} = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} \tag{13.5}$$

the game is said to have a *saddle point*. Clearly, in this case, the best course of action for P1 will be to choose strategy r while P2 will optimize her performance by choosing strategy k. As an example, and given the following payoff matrix A

$$\begin{array}{cc} & \text{Player } 2 \Rightarrow j \\ \text{Player } 1 \Rightarrow i & \begin{pmatrix} 7 & 2 & 1 \\ 2 & 2 & 3 \\ 5 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix} \end{array}$$

$\min_j a_{ij}$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 2 \end{array} \qquad \begin{array}{c} \max_i a_{ij} \\ 7 \quad 3 \quad 6 \end{array}$$

Hence,

$$a_{32} = \max_i \min_j a_{ij} = 3 \quad = \quad \min_j \max_i a_{ij} = 3$$

and this game has a saddle point. A remarkable property of pure strategy games with a saddle point is that security measures of secrecy are not necessary. In other words, any player can reveal his choice of strategy while the other player will be unable to take advantage of this information.

On the contrary, if

$$\max_i \min_j a_{ij} < \min_j \max_i a_{ij} \tag{13.6}$$

the game does not have a saddle point and the game is not stable. This event is given by the following payoff matrix A

$$\begin{array}{cc} & \text{Player } 2 \Rightarrow j \\ \text{Player } 1 \Rightarrow i & \begin{pmatrix} 7 & 4 & 1 \\ 2 & 2 & 3 \\ 5 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix} \end{array}$$

$\min_j a_{ij}$

1
2
3
2

$\max_i a_{ij}$

7 4 6

Hence,

$$\max_i \min_j a_{ij} = 3 \qquad < \qquad \min_j \max_i a_{ij} = 4$$

and this game does not have a saddle point. Both players feel that they could do better by choosing a different criterion of choosing their strategy.

In order to solve this problem, Von Neumann introduced the notion of mixed strategy and proved the famous Minimax theorem. A mixed strategy is a selection of pure strategies weighted by fixed probabilities. That is, suppose that player 1 chooses pure strategy i with probability $x_i \geq 0$ and $\sum_{i=1}^{m} x_i = 1$. This selection can be realized using an appropriate chance device. Similarly, player 2 chooses pure strategy j with probability $y_j \geq 0$ and $\sum_{j=1}^{n} y_j = 1$. This is a situation in which a player knows his/her strategy only after using a chance device, such as a dice, for example. As a consequence, we can only speak of an expected payoff for the game, stated as $E(\mathbf{x}, \mathbf{y}) = \mathbf{x}' A \mathbf{y} = \sum_{ij} x_i a_{ij} y_j$, assuming that player 1 uses mixed strategy $\mathbf{x}$ and player 2 uses mixed strategy $\mathbf{y}$. The expected payoff $E(\mathbf{x}, \mathbf{y})$ is also called tha value of the game.

The focus of the analysis, therefore, is shifted to the determination of the probability vectors $\mathbf{x}$ and $\mathbf{y}$. As before, player 1 is aware that his opponent will attempt to minimize the level of his winnings. In other words, player 2 is expected to choose a mixed strategy $\mathbf{y}$ such that, as far as she is concerned, the expected payoff (which is a cost to player 2) will turn out to be as small as possible, that is,

$$\min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y}).$$

Hence, the best course of action for player 1 will be to choose a mixed strategy $\mathbf{x}$ such that

$$V_1^* = \max_{\mathbf{x}} \min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y}). \tag{13.7}$$

Analogously, player 2 expects that player 1 will attempt to maximize his own winnings, that is, she expects that player 1 will choose a mixed strategy such that, from his point of view,

$$\max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y}).$$

Therefore, the best course of action of player 2 will be to choose a mixed strategy $\mathbf{y}$ such that

$$V_2^* = \min_{\mathbf{y}} \max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y}). \tag{13.8}$$

Von Neumann has shown that there always exist optimal mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$ such that $V_1^* = V_2^*$. This is called the fundamental theorem of two-person zero-sum games. It turns out that a two-person zero-sum game can be always formulated as a dual pair of linear programming problems. Remarkably, the primal problem corresponds to the behavior of player 1 while the dual problem expresses the behavior of player 2. This LP specification is also due to Von Neumann and it represents the most elegant proof of the fundamental theorem of two-person zero-sum games.

The justification of the LP format is based upon the observation that player 1, when choosing mixed strategy $\mathbf{x}$, can expect his payoff to be

$$\sum_{i=1}^{m} a_{ji} x_i \qquad j = 1, \ldots, n. \tag{13.9}$$

It will be in his interest, therefore, to make this amount as large as possible among the n selections available to player 2. In other words, player 1 will attempt to maximize a lower bound L on the admissible n levels of payoff accruable to him and as expressed by (13.9). Given this justification, the primal linear programming specification that will represent this type of behavior is given by the problem of finding a vector $\mathbf{x} \geq \mathbf{0}$ such that

Primal : Player 1

$$\max_{\mathbf{x}, L} L \tag{13.10}$$

subject to

$$\mathbf{s}_n L - A'\mathbf{x} \leq \mathbf{0}$$
$$\mathbf{s}_m' \mathbf{x} = 1$$

where $\mathbf{s}_n$ and $\mathbf{s}_m$ are vectors of unit elements of dimension $(n \times 1)$ and $(m \times 1)$, respectively. Vectors $\mathbf{s}_n$ and $\mathbf{s}_m$ are also called *sum vectors*. The first constraint of (13.10) states a floor (lower bound) on each possible strategy choice of player 2. Player 1 wishes to make this floor as high as possible. The second constraint of (13.10) is simply the adding-up condition of probabilities.

By selecting symbols $\mathbf{y}$ and R as dual variables of the primal constraints in (13.10), we can state the corresponding dual problem as finding a vector $\mathbf{y} \geq \mathbf{0}$ such that

Dual : Player 2

$$\min_{\mathbf{y}, R} R \tag{13.11}$$

subject to

$$\mathbf{s}_m R - A\mathbf{y} \geq \mathbf{0}$$
$$\mathbf{s}_n' \mathbf{y} = 1.$$

The dual problem (13.11) represents the optimal behavior of player 2. She will want to minimize the ceiling (upper bound) of any possible strategy chosen by player 1.

Let us assume that there exist feasible vectors $\mathbf{x}^*$ and $\mathbf{y}^*$. Then, from the primal constraints

$$\mathbf{y}^{*\prime}\mathbf{s}_n L - \mathbf{y}^{*\prime} A'\mathbf{x}^* \leq 0 \tag{13.12}$$
$$L \leq \mathbf{y}^{*\prime} A'\mathbf{x}^*.$$

Similarly, from the dual constraints

$$\mathbf{x}^{*\prime}\mathbf{s}_m R - \mathbf{x}^{*\prime} A\mathbf{y}^* \geq 0 \tag{13.13}$$
$$R \geq \mathbf{x}^{*\prime} A\mathbf{y}^*.$$

Hence, $L^* = \max L = \mathbf{x}^{*\prime} A\mathbf{y}^* = \min R = R^*$ and the value of the game is $V^* = L^* = R^*$, according to the duality theory of linear programming.

An alternative, but equivalent way, to establish the saddle-point property of a two-person zero-sum game is to derive and analyze the KKT conditions of the LP problem (13.10). In this case, the corresponding Lagrangean function, $\mathcal{L}$, is specified as

$$\mathcal{L} = L + \mathbf{y}'(A'\mathbf{x} - \mathbf{s}_n L) + R(1 - \mathbf{s}_m'\mathbf{x}) \tag{13.14}$$

with the following KKT conditions:

$$\frac{\partial \mathcal{L}}{\partial L} = 1 - \mathbf{s}_n'\mathbf{y} \leq 0 \tag{13.15}$$
$$L\frac{\partial \mathcal{L}}{\partial L} = L(1 - \mathbf{s}_n'\mathbf{y}) = 0 \tag{13.16}$$
$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = A\mathbf{y} - \mathbf{s}_m R \leq \mathbf{0} \tag{13.17}$$
$$\mathbf{x}'\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \mathbf{x}'A\mathbf{y} - \mathbf{x}'\mathbf{s}_m R = 0 \tag{13.18}$$
$$\frac{\partial \mathcal{L}}{\partial R} = 1 - \mathbf{s}_m'\mathbf{x} \geq 0 \tag{13.19}$$
$$R\frac{\partial \mathcal{L}}{\partial R} = R(1 - \mathbf{s}_m'\mathbf{x}) = 0 \tag{13.20}$$
$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}} = A'\mathbf{x} - \mathbf{s}_n L \geq \mathbf{0} \tag{13.21}$$
$$\mathbf{y}'\frac{\partial \mathcal{L}}{\partial \mathbf{y}} = \mathbf{y}'A'\mathbf{x} - \mathbf{y}'\mathbf{s}_n L = 0 \tag{13.22}$$

Using the information of KKT conditions (13.16), (13.18), (13.20) and (13.22) we reach the conclusion that, for $(L \neq 0)$ and $(R \neq 0)$, $L = \mathbf{y}'A'\mathbf{x} = R$, as expected.

The condition that $(L \neq 0)$ and $(R \neq 0)$ can always be fulfilled. In fact, the value of the game can always be regarded as a positive amount. This means that the solution of a game, represented by vectors $\mathbf{x}^*$ and $\mathbf{y}^*$, is the same when a payoff matrix A with positive and negative elements is augmented by a matrix C with constant positive elements.

To demonstrate this assertion, let the $(m \times n)$ matrix C be defined by a constant scalar k and vectors $\mathbf{s}_m$ and $\mathbf{s}_n$ such that $C = k\mathbf{s}_m\mathbf{s}_n'$. This means that $C\mathbf{y} = k\mathbf{s}_m\mathbf{s}_n'\mathbf{y} = k\mathbf{s}_m$ and $\mathbf{x}'C\mathbf{y} = k\mathbf{x}'\mathbf{s}_m = k$. Now we may augment the original payoff matrix A by the matrix C to make any payoff element a strictly positive amount. The corresponding LP problem now becomes

Primal : Player 1

$$\max_{\mathbf{x},L} L \tag{13.23}$$

subject to
$$\mathbf{s}_n L - (A' + C')\mathbf{x} \leq \mathbf{0}$$
$$\mathbf{s}_m'\mathbf{x} = 1.$$

We will show that the primal problem (13.23) is equivalent to primal problem (13.10) and, therefore, their solutions, in terms of mixed strategies $(\mathbf{x}, \mathbf{y})$, are identical. Notice that the structure of matrix C allows the redefinition of the primal constraints in (13.23) as follows

Primal : Player 1

$$\max_{\mathbf{x},L} L \tag{13.24}$$

subject to
$$\mathbf{s}_n L - A'\mathbf{x} \leq k\mathbf{s}_n$$
$$\mathbf{s}_m'\mathbf{x} = 1.$$

The dual specification of problem (13.24) is

Dual : Player 2

$$\min_{\mathbf{y},R} R + k \tag{13.25}$$

subject to
$$\mathbf{s}_m R - A\mathbf{y} \geq \mathbf{0}$$
$$\mathbf{s}_n'\mathbf{y} = 1.$$

Recall that the symbol k is a constant scalar and, thus, does not affect the optimization process. This dual problem (13.25) has exactly the same structure of problem (13.11) which, in turn, corresponds to the dual specification of problem (13.10). Hence, the primal problem (13.23), with regard to mixed strategies, is equivalent to problem (13.10), as asserted.

Two-Person Non-Zero-Sum Games

Two-person non-zero-sum games are also called *bimatrix* games because they are characterized by two distinct payoff matrices, A and B, one for each player. Both matrices A and B have m rows and n columns. Elements a_{ij} and b_{ij} are the payoffs to player 1 and player 2, respectively, when player 1 plays pure strategy $i, i = 1, \ldots, m$ and player 2 plays pure strategy $j, j = 1, \ldots, n$.

Mixed strategies are pure strategies played according to probability distributions denoted by the vector $\mathbf{x}$ for player 1 and by the vector $\mathbf{y}$ for player 2. Hence, vectors $\mathbf{x}$ and $\mathbf{y}$ have m and n nonnegative elements, respectively, adding up to unity. Therefore, in matrix form, a pair of mixed strategies $(\mathbf{x}, \mathbf{y})$ is written as

$$\mathbf{x}'\mathbf{s}_m = 1, \quad \mathbf{x} \geq \mathbf{0} \qquad\qquad \mathbf{y}'\mathbf{s}_n = 1, \quad \mathbf{y} \geq \mathbf{0} \tag{13.26}$$

with expected payoffs to player 1 and player 2, respectively,

$$\mathbf{x}'A\mathbf{y} \qquad\qquad \mathbf{x}'B\mathbf{y}. \tag{13.27}$$

As in the case of a two-person zero-sum game discussed in the preceding section, we can assume that all the elements of matrices A and B are strictly positive without affecting the vectors of mixed strategies, as Lemke and Howson have demonstrated. A bimatrix game is completely specified by the pair of matrices $[A, B]$.

The solution of a bimatrix game consists in finding the vectors of mixed strategies $(\mathbf{x}, \mathbf{y})$ such that they lead to an equilibrium of the game. A characteristic of a bimatrix game is that it cannot be expressed as an optimization problem. In other words, there is no dual pair of optimization problems that can be used to solve a bimatrix game as there is for a two-person zero-sum game.

The following discussion is taken from Lemke. Nash has demonstrated that an *equilibrium point* for the game $[A, B]$ is a pair of mixed strategies $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that, for all pairs of mixed strategies $(\mathbf{x}, \mathbf{y})$ satisfying (13.26),

$$\bar{\mathbf{x}}'A\bar{\mathbf{y}} \leq \mathbf{x}'A\bar{\mathbf{y}}, \qquad \text{and} \qquad \bar{\mathbf{x}}'B\bar{\mathbf{y}} \leq \bar{\mathbf{x}}'B\mathbf{y}. \tag{13.28}$$

The interpretation of $\mathbf{x}'A\mathbf{y}$ and $\mathbf{x}'B\mathbf{y}$ is the expected loss to player 1 and player 2, respectively, if player 1 plays according to probabilities $\mathbf{x}$ and player 2 plays according to probabilities $\mathbf{y}$. Each player attempts to minimize his own expected loss under the assumption that each player knows the equilibrium strategies of his opponent.

To further analyze system (13.28), let $\mathbf{e}_j$ be a unit vector with all components being equal to zero and only the j-th component being equal to 1. It is clear that $\mathbf{e}_j$ is a mixed strategy called *pure strategy*. Hence, system (13.28) holds if and only if it holds for all pure strategies $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}_i, \mathbf{e}_j), i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then, in vector form, system (13.28) can be expressed as

$$\begin{array}{lcl} (\bar{\mathbf{x}}' A \bar{\mathbf{y}}) & \leq & \mathbf{e}_1' A \bar{\mathbf{y}} \\ (\bar{\mathbf{x}}' A \bar{\mathbf{y}}) & \leq & \mathbf{e}_2' A \bar{\mathbf{y}} \\ & \vdots & \\ (\bar{\mathbf{x}}' A \bar{\mathbf{y}}) & \leq & \mathbf{e}_m' A \bar{\mathbf{y}} \end{array} \qquad \begin{array}{lcl} (\bar{\mathbf{x}}' B \bar{\mathbf{y}}) & \leq & \bar{\mathbf{x}}' B \mathbf{e}_1 \\ (\bar{\mathbf{x}}' B \bar{\mathbf{y}}) & \leq & \bar{\mathbf{x}}' B \mathbf{e}_2 \\ & \vdots & \\ (\bar{\mathbf{x}}' B \bar{\mathbf{y}}) & \leq & \bar{\mathbf{x}}' B \mathbf{e}_n \end{array} \tag{13.29}$$

or, in more compact form and with reference only to the first system based on the A matrix,

$$(\bar{\mathbf{x}}' A \bar{\mathbf{y}}) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix} A \bar{\mathbf{y}}$$

and, furthermore,

$$(\bar{\mathbf{x}}' A \bar{\mathbf{y}}) \mathbf{s}_m \leq A \bar{\mathbf{y}} \tag{13.30}$$

A similar discussion, with respect to the system based on the B matrix, leads to the following relation

$$(\bar{\mathbf{x}}' B \bar{\mathbf{y}}) \mathbf{s}_n' \leq \bar{\mathbf{x}}' B \tag{13.31}$$

The goal is now to restate systems (13.30) and (13.31) in the form of a linear complementarity problem. Therefore, the following constraints

$$\phi_1 \mathbf{s}_m \leq A \bar{\mathbf{y}} \qquad \bar{\mathbf{x}}'(A \bar{\mathbf{y}} - \phi_1 \mathbf{s}_m) = 0 \tag{13.32}$$

$$\phi_2 \mathbf{s}_n \leq B' \bar{\mathbf{x}} \qquad \bar{\mathbf{y}}'(B' \bar{\mathbf{x}} - \phi_2 \mathbf{s}_n) = 0 \tag{13.33}$$

where $\phi_1 = (\bar{\mathbf{x}}' A \bar{\mathbf{y}}) > 0$ and $\phi_2 = (\bar{\mathbf{x}}' B \bar{\mathbf{y}}) > 0$, are equivalent to (13.30) and (13.31).

Finally, by defining new vector variables $\mathbf{x} = \bar{\mathbf{x}}/\phi_2$ and $\mathbf{y} = \bar{\mathbf{y}}/\phi_1$ and introducing slack vectors $\mathbf{v}$ and $\mathbf{u}$, system (13.32) and (13.33) can be rewritten in the form of a LC problem $(M, \mathbf{q})$

$$\begin{bmatrix} 0 & A \\ B' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} -\mathbf{s}_m \\ -\mathbf{s}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} \geq \mathbf{0}, \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \mathbf{0} \tag{13.34}$$

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}' \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = 0 \tag{13.35}$$

$$\text{where } M = \begin{bmatrix} 0 & A \\ B' & 0 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -\mathbf{s}_m \\ -\mathbf{s}_n \end{bmatrix}.$$

A solution $(\mathbf{x}, \mathbf{y})$ to the LC problem (13.34) and (13.35) can be transformed into a solution of the original bimatrix problem (13.30) and (13.31) by the following computations:

$$A\mathbf{y} \geq \mathbf{s}_m \quad \rightarrow \quad \mathbf{x}' A\mathbf{y} = \mathbf{x}'\mathbf{s}_m \quad \rightarrow \quad \frac{\bar{\mathbf{x}}'}{\phi_2}\frac{A}{}\frac{\bar{\mathbf{y}}}{\phi_1} = \mathbf{x}'\mathbf{s}_m \tag{13.36}$$

$$\frac{1}{\phi_2} = \mathbf{x}'\mathbf{s}_m \quad \rightarrow \quad \phi_2 = \frac{1}{\mathbf{x}'\mathbf{s}_m}$$

since $\phi_1 = \bar{\mathbf{x}}' A\bar{\mathbf{y}}$ by definition. Therefore, a solution for the bimatrix game will be achieved by retracing the steps from the definitions given above: $\bar{\mathbf{x}} = \mathbf{x}\phi_2 = \mathbf{x}/\mathbf{x}'\mathbf{s}_m$. Analogous computations will lead to the second part of the solution, that is, $\bar{\mathbf{y}} = \mathbf{y}\phi_1 = \mathbf{y}/\mathbf{y}'\mathbf{s}_n$.

Algorithm for Solving a Bimatrix Game

The description of this algorithm, as given by Lemke (p. 106), is reproduced below. The bimatrix game, as articulated in (13.34) and (13.35), may be expressed in the form:

$$\mathbf{I}. \qquad \mathbf{u} = -\mathbf{s}_n + B'\mathbf{x}, \qquad \mathbf{u} \geq \mathbf{0}, \qquad \mathbf{x} \geq \mathbf{0} \tag{13.37}$$

$$\mathbf{II}. \qquad \mathbf{v} = -\mathbf{s}_m + A\mathbf{y}, \qquad \mathbf{v} \geq \mathbf{0}, \qquad \mathbf{y} \geq \mathbf{0} \tag{13.38}$$

together with the complementary condition

$$\mathbf{u}'\mathbf{y} = 0 \qquad \text{and} \qquad \mathbf{v}'\mathbf{x} = 0. \tag{13.39}$$

Hence, $(v_i, x_i), i = 1, \ldots, m$ and $(u_j, y_j), j = 1, \ldots, n$ are complementary pairs, that is, $v_i x_i = 0, i = 1, \ldots, m$ and $u_j y_j = 0, j = 1, \ldots, n$.

The two systems **I** and **II** are, seemingly, disjoint. The complementary pivot algorithm consists of the following sequence:

1. *First pivot.* In system **I**, increase x_1 to obtain a feasible solution of system **I**. That is, pivot on the pair (u_r, x_1), with index r automatically defined by feasibility conditons.
2. *Second pivot.* Having determined the index r on the first pivot, in system **II** increase y_r (complement of u_r) to obtain the initial feasibility of system **II**. Let (v_s, y_r) be the pivot pair.

Then, if $s = 1$, the resulting basic points satisfy (13.39), and the pivoting terminates. If not, it continues.

The algorithm consists of alternating pivots on systems **I** and **II**, always increasing the complement of the variable which, on the immediately-previous pivot, became nonbasic. This pivoting scheme is automatic. Furthermore, the algorithm will always terminate in a complementary solution (Lemke, p. 107, Theorem III).

It is convenient to restate the complementary pivot algorithm for a bimatrix game in a compact form as allowed by the LC problem. This is the form adopted by the computer program described in chapter 15, problem 7. Hence,

$$\mathbf{w} = \mathbf{q} + M\mathbf{z}, \qquad \mathbf{w} \geq \mathbf{0}, \qquad \mathbf{z} \geq \mathbf{0} \tag{13.40}$$

together with the complementary condition

$$\mathbf{z}'\mathbf{w} = 0 \tag{13.41}$$

where

$$M = \begin{bmatrix} 0 & A \\ B' & 0 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -\mathbf{s}_m \\ -\mathbf{s}_n \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}.$$

The initial basis of system (13.40) is the identity matrix associated with the slack vector $\mathbf{w}$. Thus, $\mathbf{z} = \mathbf{0}$ and the complementary condition (13.41) is satisfied. Notice that, at this initial step, the indexes of the basic variables constitute an uninterrupted series from 1 to $(m+n)$. The first step of the complementary pivot algorithm, then, consists in increasing the variable z_1, by introducing the corresponding vector into the new basis. Presumably, the vector associated with variable w_r will be eliminated from the current basis. Now, if $r \neq 1$, the complementary condition is violated because $z_1 w_1 \neq 0$. At this stage, the basic index series is interrupted because variables z_1 and its complement w_1 have become basic variables.

The pivoting scheme of the algorithm continues by choosing to increase the complementary variable (z_r, in this case) and keeping track of the variable which is reduced to zero. The algorithm terminates when either the variable w_1 or z_1 is reduced to zero, following the automatic pivoting scheme. At this stage, the index series of basic variables has become uninterrupted, again.

A Numerical Example of a Bimatrix Game

The numerical example of a two-person non-zero-sum game is defined by the following two payoff matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}.$$

Hence, the payoffs to Player 1 and Player 2 are, respectively:

$$\phi_1 = \bar{\mathbf{x}}'A\bar{\mathbf{y}}, \qquad \phi_2 = \bar{\mathbf{y}}'B'\bar{\mathbf{x}}$$

where $\bar{\mathbf{x}} \geq \mathbf{0}, \bar{\mathbf{y}} \geq \mathbf{0}, \mathbf{s}_m'\bar{\mathbf{x}} = 1$ and $\mathbf{s}_n'\bar{\mathbf{y}} = 1$.

The transformation of the game into a LC problem follows the structure of relations (13.32) and (13.33), where $\mathbf{x} = \bar{\mathbf{x}}/\phi_2$ and $\mathbf{y} = \bar{\mathbf{y}}/\phi_1$. The corresponding LC problem, therefore, takes on the following configuration, according to relations (13.34) and (13.35):

$$M = \begin{bmatrix} & & & 1 & 4 \\ & & & 3 & 1 \\ & & & 2 & 2 \\ 3 & 1 & 4 & & \\ 2 & 2 & 1 & & \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix}.$$

In order to simplify the illustration and the understanding of the complementary pivot algorithm, we will use the compact form in terms of vectors $\mathbf{z}$ and $\mathbf{w}$ rather than the more articulated specification illustrated in relations (13.37), (13.38) and (13.39). The recovery of the information in terms of vectors $\mathbf{x}, \mathbf{y}, \mathbf{u}$ and $\mathbf{v}$ is straightforward.

In the following three tableaux, the specification of the LC problem assumes the following rearrangement: $\mathbf{w} - M\mathbf{z} = \mathbf{q}$. Blank entries correspond to zero values.

$$\begin{array}{c} \\ T_0 \\ \begin{bmatrix} \\ \\ \\ \frac{-3}{2} \\ \frac{-1}{2} \end{bmatrix} \end{array}
\begin{array}{c} \downarrow \text{initial step} \\
\begin{array}{ccccccccccc} w_1 & w_2 & w_3 & w_4 & w_5 & z_1 & z_2 & z_3 & z_4 & z_5 & \mathbf{q} \end{array} \\
\begin{bmatrix} 1 & & & & & & & & -1 & -4 & -1 \\ & 1 & & & & & & & -3 & -2 & -1 \\ & & 1 & & & & & & -2 & -2 & -1 \\ & & & 1 & & -3 & -1 & -4 & & & -1 \\ & & & & 1 & (-2) & -2 & -1 & & & -1 \end{bmatrix} \end{array}
\begin{array}{l} \\ BI \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \rightarrow \end{array}$$

$$\begin{array}{c} T_1 \\ \begin{bmatrix} -4 \\ -1 \\ -2 \\ \\ \\ \end{bmatrix} \end{array}
\begin{array}{c} \downarrow \text{complement of } \mathrm{w}_5 \\
\begin{bmatrix} 1 & & & & & & & & -1 & -4 & -1 \\ & 1 & & & & & & & -3 & (-1) & -1 \\ & & 1 & & & & & & -2 & -2 & -1 \\ & & & 1 & \frac{-3}{2} & 0 & 2 & \frac{-5}{2} & & & \frac{1}{2} \\ & & & & \frac{-1}{2} & 1 & 1 & \frac{1}{2} & & & \frac{1}{2} \end{bmatrix} \end{array}
\begin{array}{l} \\ w_1 \\ w_2 \rightarrow \\ w_3 \\ w_4 \\ z_1 \end{array}$$

$$\begin{array}{c} T_2 \\ \begin{bmatrix} \\ \\ \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \end{array}
\begin{array}{c} \downarrow \text{complement of } \mathrm{w}_2 \\
\begin{bmatrix} 1 & -4 & & & & & & & 11 & 0 & 3 \\ & -1 & & & & & & & 3 & 1 & 1 \\ & -2 & 1 & & & & & & 4 & 0 & 1 \\ & & & 1 & \frac{-3}{2} & 0 & (2) & \frac{-5}{2} & & & \frac{1}{2} \\ & & & & \frac{-1}{2} & 1 & 1 & \frac{1}{2} & & & \frac{1}{2} \end{bmatrix} \end{array}
\begin{array}{l} \\ w_1 \\ z_5 \\ w_3 \\ w_4 \rightarrow \\ z_1 \end{array}$$

The initial step of the complementary pivot algorithm, in its variant for dealing with a bimatrix game, is to turn all the values of the $\mathbf{q}$ vector into nonnegative elements in order to achieve feasibility of the solution. Given the seemingly disjoint nature of the overall system of equations, as stated in relations (13.37) and (13.38), this goal is achieved in the third tableau, after two iterations that are guided by a special choice of the pivot element.

In the initial tableau, the basis variables are all slack variables, that is, $\mathbf{w} = \mathbf{q}$ and $\mathbf{z} = \mathbf{0}$. The complementarity condition $\mathbf{w}'\mathbf{z} = 0$ is satified, but the solution is not feasible since all the elements of the $\mathbf{q}$ vector are negative. Notice that the index series of the basic variables, $w_j, j = 1, 2, 3, 4, 5$, is uninterrupted. The algorithm begins by introducing the column vector associated with the variable z_1 into the basis. The choice of z_1 is arbitrary but, once esecuted, the selection of all the successive column vectors to enter the basis is automatically determined by the complementary condition.

In the first tableau, the choice of the pivot element is determined by the following criterion: pivot $= \max(q_i/m_{i1}), i = 1, \ldots, N$ for $m_{i1} < 0$ and where m_{i1} are the coefficients of the first column in the $[-M]$ matrix. In the first tableau, the pivot corresponds to (-2) and it is indicated by the inclusion of this value in parentheses. The updating of the $\mathbf{q}$ vector and the basis inverse eliminates the w_5 variable from the set of basic variables and replaces it with the z_1 variable (second tableau). Hence, the next candidate vector to enter the basis is associated with variable z_5, the complement of w_5. The updating of each tableau is done by premultiplying the entire current tableau by the transformation matrix T_k, where k is the iteration index, as explained in chapter 4.

The second step of the algorithm follows the same criterion in the choice of the pivot element and achieves the feasibility of the solution in the third tableau. Notice that feasibility of the solution has been achieved at the (temporary) expense of the complementarity condition. That is, in the third tableau (as in the second tableau), the complementarity condition is violated because $w_1 z_1 \neq 0$, indicating precisely that the solution is *almost* complementary. A visual way to assess the violation of the complementarity condition is given by the index series of the basic variables which is now interrupted, with a gap between indices. For example, in the third tableau, the series of the basic variables (under the BI heading) is $1, 5, 3, 4, 1$, missing the index 2.

The algorithm proceeds in the next three tableaux with the goal of maintaining the feasibilility of the solution while striving to achieve its full complementarity. Beginning with the third tableau, the choice of the pivot element must be modified in order to maintain the feasibility of the solution. From now on, the choice of the pivot is determined as pivot $= \min(q_i/m_{ir}), i = 1, \ldots, N$, for $m_{ir} > 0$, where the r index refers to the vector selected to enter the basis.

From the preceding third tableau $\qquad\qquad \downarrow$ complement to w_4

$$
\begin{array}{c|ccccc|ccccc|c|l}
T_3 & w_1 & w_2 & w_3 & w_4 & w_5 & z_1 & z_2 & z_3 & z_4 & z_5 & \mathbf{q} & BI \\
-\frac{11}{4} & 1 & -4 & & & & & & & 11 & 0 & 3 & w_1 \\
-\frac{3}{4} & & -1 & & & & & & & 3 & 1 & 1 & z_5 \\
\frac{1}{4} & & -2 & 1 & & & & & & (4) & 0 & 1 & w_3 \rightarrow \\
 & & & & \frac{1}{2} & \frac{-3}{4} & 0 & 1 & \frac{-5}{4} & & & \frac{1}{4} & z_2 \\
 & & & & \frac{-1}{2} & \frac{1}{4} & 1 & 0 & \frac{7}{4} & & & \frac{1}{4} & z_1
\end{array}
$$

T_4 $\qquad\qquad \downarrow$ complement of w_3

$$
\begin{array}{c|ccccc|ccccc|c|l}
 & 1 & \frac{3}{2} & \frac{-11}{4} & & & & & & 0 & 0 & \frac{1}{4} & w_1 \\
 & & \frac{1}{2} & \frac{-3}{4} & & & & & & 0 & 1 & \frac{1}{4} & z_5 \\
 & & \frac{-1}{2} & \frac{1}{4} & & & & & & 1 & 0 & \frac{1}{4} & z_4 \\
\frac{5}{7} & & & & \frac{1}{2} & \frac{-3}{4} & 0 & 1 & \frac{-5}{4} & & & \frac{1}{4} & z_2 \\
\frac{4}{7} & & & & \frac{-1}{2} & \frac{1}{4} & 1 & 0 & (\frac{7}{4}) & & & \frac{1}{4} & z_1 \rightarrow
\end{array}
$$

final tableau : solution

$$
\begin{array}{ccccc|ccccc|c|l}
1 & \frac{3}{2} & \frac{-11}{4} & & & & & & 0 & 0 & \frac{1}{4} & w_1 \\
 & \frac{1}{2} & \frac{-3}{4} & & & & & & 0 & 1 & \frac{1}{4} & z_5 \\
 & \frac{-1}{2} & \frac{1}{4} & & & & & & 1 & 0 & \frac{1}{4} & z_4 \\
 & & & \frac{1}{7} & \frac{-4}{7} & \frac{5}{7} & 1 & 0 & & & \frac{3}{7} & z_2 \\
 & & & \frac{-2}{7} & \frac{1}{7} & \frac{4}{7} & 0 & 1 & & & \frac{1}{7} & z_3
\end{array}
$$

The solution of the bimatrix game is achieved in the last tableau. Notice that the index series of the basic variables has been reconstituted with indices $1, 5, 4, 2, 3$, without any gap. This is a visual indication that the current solution is both feasible and complementary.

The reading of the LC problem's solution from the final tableau and the conversion of this solution into a solution of the bimatrix game proceeds as follow.

$$
\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} z_1 \equiv x_1 \\ z_2 \equiv x_2 \\ z_3 \equiv x_3 \\ \\ z_4 \equiv y_1 \\ z_5 \equiv y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/7 \\ 1/7 \\ \\ 1/4 \\ 1/4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} w_1 \equiv v_1 \\ w_2 \equiv v_2 \\ w_3 \equiv v_3 \\ \\ w_4 \equiv u_1 \\ w_5 \equiv u_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix}
$$

From the discussion of the previous section, the solution of the bimatrix game, corresponding to the mixed strategies of player 1 and player 2, is extracted from the relations: $\bar{\mathbf{x}} = \mathbf{x}/\mathbf{s}_m'\mathbf{x}$ and $\bar{\mathbf{y}} = \mathbf{y}/\mathbf{s}_n'\mathbf{y}$. Hence,

$$\mathbf{s}_m'\mathbf{x} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\3/7\\1/7\end{bmatrix} = \frac{4}{7}, \qquad \mathbf{s}_n'\mathbf{y} = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1/4\\1/4\end{bmatrix} = \frac{2}{4}.$$

Therefore, the vectors of mixed strategies for player 1 and player 2, respectively, are

$$\textbf{Player 1}: \qquad \bar{\mathbf{x}} = \mathbf{x}/\mathbf{s}_m'\mathbf{x} = \begin{bmatrix}0\\3/7\\1/7\end{bmatrix}\Big/\frac{4}{7} = \begin{bmatrix}0\\3/4\\1/4\end{bmatrix}$$

$$\textbf{Player 2}: \qquad \bar{\mathbf{y}} = \mathbf{y}/\mathbf{s}_n'\mathbf{y} = \begin{bmatrix}1/4\\1/4\end{bmatrix}\Big/\frac{2}{4} = \begin{bmatrix}1/2\\1/2\end{bmatrix}.$$

Finally, the payoff values are

$$\textbf{Player 1}: \qquad \phi_1 = \bar{\mathbf{x}}'A\bar{\mathbf{y}} = \begin{bmatrix}0 & \frac{3}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1 & 4\\3 & 1\\2 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \frac{1}{\mathbf{y}'\mathbf{s}_n} = 2$$

$$\textbf{Player 2}: \qquad \phi_2 = \bar{\mathbf{x}}'B\bar{\mathbf{y}} = \begin{bmatrix}0 & \frac{3}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 2\\4 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \frac{1}{\mathbf{x}'\mathbf{s}_m} = \frac{7}{4}.$$

This concludes the discussion of the numerical example of a bimatrix game.

Lemke's Complementary Pivot Algorithm for bimatrix games requires that the payoff matrices A and B have all positive elements. This requirement does not affect the solution of a bimatrix game. This statement can be demonstrated by the following reasoning.

Let us suppose that the payoff matrices A, B contain some negative elements and that there exists an equilibrium point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ for the game $[A, B]$.

Now, let $C = \mathbf{s}_m\mathbf{s}_n'$ be the matrix with all unitary elements and let the scalar k be large enough so that the matrices $(kC+A) > 0$ and $(kC'+B') > 0$, that is, they have all positive elements. Consider now the solution $\mathbf{x}, \mathbf{y}$ to the following relations

$$(kC + A)\mathbf{y} \geq \mathbf{s}_m, \quad \mathbf{y} \geq \mathbf{0} \qquad \text{and} \qquad \mathbf{x}'[(kC + A)\mathbf{y} - \mathbf{s}_m] = 0 \tag{13.42}$$

$$(kC' + B')\mathbf{x} \geq \mathbf{s}_n, \quad \mathbf{x} \geq \mathbf{0} \qquad \text{and} \qquad \mathbf{y}'[(kC' + B')\mathbf{x} - \mathbf{s}_n] = 0. \tag{13.43}$$

The solution $\mathbf{x}, \mathbf{y}$ does not, in general, satisfy the adding-up conditions $\mathbf{x}'\mathbf{s}_m = 1$ and $\mathbf{y}'\mathbf{s}_n = 1$ since no such constraints are involved in relations (13.42) and (13.43).

Now, consider relation (13.42). By expanding the complementary condition we write

$$\begin{aligned} \mathbf{x}'[(kC + A)\mathbf{y} - \mathbf{s}_m] &= 0 \\ k\mathbf{x}'C\mathbf{y} + \mathbf{x}'A\mathbf{y} - \mathbf{x}'\mathbf{s}_m &= 0 \\ \frac{k\mathbf{x}'C\mathbf{y}}{\mathbf{x}'\mathbf{s}_m\mathbf{s}_n'\mathbf{y}} + \frac{\mathbf{x}'}{\mathbf{x}'\mathbf{s}_m}A\frac{\mathbf{y}}{\mathbf{s}_n'\mathbf{y}} - \frac{1}{\mathbf{s}_n'\mathbf{y}} &= 0 \\ k + \bar{\mathbf{x}}'A\bar{\mathbf{y}} - \frac{1}{\mathbf{s}_n'\mathbf{y}} &= 0 \end{aligned} \tag{13.44}$$

where

$$\bar{\mathbf{x}} = \frac{\mathbf{x}'}{\mathbf{x}'\mathbf{s}_m} \qquad \text{and} \qquad \bar{\mathbf{y}} = \frac{\mathbf{y}'}{\mathbf{y}'\mathbf{s}_n} \tag{13.45}$$

which are the same relations presented in (13.36).

The correspondence between the payoff is

$$\bar{\mathbf{x}}'A\bar{\mathbf{y}} = \frac{1}{\mathbf{s}_n'\mathbf{y}} - k. \tag{13.46}$$

A similar calculation can be done for relation (13.43), with the result that

$$\bar{\mathbf{y}}'B'\bar{\mathbf{x}} = \frac{1}{\mathbf{s}_m'\mathbf{x}} - k. \tag{13.47}$$

This discussion establishes that the augmentation of the payoff matrices A and B by a constant matrix kC, as defined above, does not affect the equilibrium point of the bimatrix game.

Finally, what happens, or may happen, if there are ties in the columns of either the payoff matrix A or B'? Ties may cause the bimatrix game to be degenerate in the sense that the Complementary Pivot Algorithm may cycle between two successive entries/exit operations without reaching an equilibrium point. Without entering in the technical discussion of ties and degeneracy, we point out that the presence of degeneracy may signal the presence of multiple equilibrium points. A simple procedure for breaking ties is to add a small arbitrary positive constant to one of the tie elements. This is the procedure implemented in the Lemke computer program presented in chapter 16.

Maximizing Expected Gain

The preceding discussion of a bimatrix game, including the numerical example, was conducted under the assumption that players wish to minimize their expected loss. If the game involves the maximization of expected gain, the corresponding Nash equilibrium for the game $[A, B]$ must be stated as a pair of mixed strategies $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that, for all pairs of mixed strategies $(\mathbf{x}, \mathbf{y})$ satisfying (13.26),

$$\bar{\mathbf{x}}'A\bar{\mathbf{y}} \geq \mathbf{x}'A\bar{\mathbf{y}}, \qquad \text{and} \qquad \bar{\mathbf{x}}'B\bar{\mathbf{y}} \geq \bar{\mathbf{x}}'B\mathbf{y}. \tag{13.48}$$

The interpretation of $\mathbf{x}'A\mathbf{y}$ and $\mathbf{x}'B\mathbf{y}$ is now the expected gain to player 1 and player 2, respectively, if player 1 plays according to probabilities $\mathbf{x}$ and player 2 plays according to probabilities $\mathbf{y}$. Each player attempts to maximize his own expected gain under the assumption that each player knows the equilibrium strategies of his opponent.

The Complementary Pivot Algorithm, however, must be implemented as presented in the preceding sections also for this form of the bimatrix game. Let us suppose, therefore, that the same matrices A and B used for the above numerical example (with all positive elements) are now the payoff matrices for a game formulated as in (13.48). In order to reformulate the game (13.48) in the form suitable for submission to the Complementary Pivot Algorithm it is necessary to invert the inequalities of (13.48) and write

$$\bar{\mathbf{x}}'(-A)\bar{\mathbf{y}} \leq \mathbf{x}'(-A)\bar{\mathbf{y}}, \qquad \text{and} \qquad \bar{\mathbf{x}}'(-B)\bar{\mathbf{y}} \leq \bar{\mathbf{x}}'(-B)\mathbf{y}. \tag{13.49}$$

Furthermore, we must add a suitable positive factor, k, to each element of the matrices $(-A)$ and $(-B)$ so that the resulting matrices have all positive elements, that is, $A^* = (k\mathbf{s}_m\mathbf{s}_n' - A) > 0$ and $B^* = (k\mathbf{s}_m\mathbf{s}_n' - B) > 0$. Application of this transformation to the matrices A and B used in the previous numerical example results in the following matrices A^* and B^*:

$$A^* = \begin{bmatrix} 5 & 5 \\ 5 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 4 \\ 3 & 3 \end{bmatrix}$$

$$B^* = \begin{bmatrix} 5 & 5 \\ 5 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 1 & 4 \end{bmatrix}$$

where the factor, $k = 5$, was chosen for implementing the positivity of all the elements of matrices A^* and B^*.

The solution of a bimatrix game $[A, B]$ in the form of (13.48), requiring the maximization of expected gain by each player, is thus carried out using matrices A^* and B^* and the Complementary Pivot Algorithm as illustrated in the following tableaux.

↓ initial step (column z_1)

T_0

	w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	$\mathbf{q}$	BI
	1								-4	-1	-1	w_1
		1							-2	-4	-1	w_2
			1						-3	-3	-1	w_3
$\frac{-1}{2}$				1		(-2)	-4	-1			-1	w_4 →
$\frac{-3}{2}$					1	-3	-3	-4			-1	w_5

T_1 ↓ complement of w_4 (column z_4)

	w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	$\mathbf{q}$	BI
-2	1								-4	-1	-1	w_1
$\frac{-1}{2}$		1							(-2)	-4	-1	w_2 →
$\frac{-3}{2}$			1						-3	-3	-1	w_3
				$\frac{-1}{2}$		1	2	$\frac{1}{2}$			$\frac{1}{2}$	z_1
				$\frac{-3}{2}$	1		3	$\frac{-5}{2}$			$\frac{1}{2}$	w_5

T_2 ↓ complement of w_2 (column z_2)

	w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	$\mathbf{q}$	BI
	1	-2							0	7	1	w_1
		$\frac{-1}{2}$							1	2	$\frac{1}{2}$	z_4
		$\frac{-3}{2}$	1						0	3	$\frac{1}{2}$	w_3
$\frac{-2}{3}$				$\frac{-1}{2}$		1	2	$\frac{1}{2}$			$\frac{1}{2}$	z_1
$\frac{1}{3}$				$\frac{-3}{2}$	1		(3)	$\frac{-5}{2}$			$\frac{1}{2}$	w_5 →

T_3 complement of w_5 ↓ (column z_5)

	w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	$\mathbf{q}$	BI
$\frac{1}{7}$	1	-2							0	(7)	1	w_1 →
$\frac{-2}{7}$		$\frac{-1}{2}$							1	2	$\frac{1}{2}$	z_4
$\frac{-3}{7}$		$\frac{-3}{2}$	1						0	3	$\frac{1}{2}$	w_3
				$\frac{1}{2}$	$\frac{-2}{3}$	1		$\frac{13}{6}$			$\frac{1}{6}$	z_1
				$\frac{-1}{2}$	$\frac{1}{3}$		1	$\frac{-5}{6}$			$\frac{1}{6}$	z_2

final tableau : solution

w_1	w_2	w_3	w_4	w_5	z_1	z_2	z_3	z_4	z_5	$\mathbf{q}$	BI
$\frac{1}{7}$	$\frac{-2}{7}$							0	1	$\frac{1}{7}$	z_5
$\frac{-2}{7}$	$\frac{1}{14}$							1	0	$\frac{3}{14}$	z_4
$\frac{-3}{7}$	$\frac{-9}{14}$	1						0	0	$\frac{1}{14}$	w_3
			$\frac{1}{2}$	$\frac{-2}{3}$	1		$\frac{13}{6}$			$\frac{1}{6}$	z_1
			$\frac{-1}{2}$	$\frac{1}{3}$		1	$\frac{-5}{6}$			$\frac{1}{6}$	z_2

The solution of the bimatrix game (13.48) is achieved in the last tableau. Notice that the index series of the basic variables has been reconstituted with indices $5, 4, 3, 1, 2$, without any gap. This is a visual indication that the current solution is both feasible and complementary.

The reading of the LC problem's solution from the final tableau and the conversion of this solution into a solution of the bimatrix game proceeds as follow.

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} z_1 \equiv x_1 \\ z_2 \equiv x_2 \\ z_3 \equiv x_3 \\ \\ z_4 \equiv y_1 \\ z_5 \equiv y_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \\ 0 \\ \\ 3/14 \\ 2/14 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} w_1 \equiv v_1 \\ w_2 \equiv v_2 \\ w_3 \equiv v_3 \\ \\ w_4 \equiv u_1 \\ w_5 \equiv u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/14 \\ \\ 0 \\ 0 \end{bmatrix}$$

From the discussion of previous sections, the solution of the bimatrix game, corresponding to the mixed strategies of player 1 and player 2, is extracted from the relations: $\bar{\mathbf{x}} = \mathbf{x}/\mathbf{s}_m'\mathbf{x}$ and $\bar{\mathbf{y}} = \mathbf{y}/\mathbf{s}_n'\mathbf{y}$. Hence,

$$\mathbf{s}_m'\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/6 \\ 1/6 \\ 0 \end{bmatrix} = \frac{2}{6}, \qquad \mathbf{s}_n'\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3/14 \\ 2/14 \end{bmatrix} = \frac{5}{14}.$$

Therefore, the vectors of mixed strategies for player 1 and player 2, respectively, are

$$\textbf{Player 1}: \qquad \bar{\mathbf{x}} = \mathbf{x}/\mathbf{s}_m'\mathbf{x} = \begin{bmatrix} 1/6 \\ 1/6 \\ 0 \end{bmatrix} \Big/ \frac{2}{6} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\textbf{Player 2}: \qquad \bar{\mathbf{y}} = \mathbf{y}/\mathbf{s}_n'\mathbf{y} = \begin{bmatrix} 3/14 \\ 2/14 \end{bmatrix} \Big/ \frac{5}{14} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}.$$

Finally, the payoff values are

$$\textbf{Player 1}: \qquad \phi_1 = \bar{\mathbf{x}}'A\bar{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \\ \frac{2}{5} \end{bmatrix} = \frac{1}{\mathbf{y}'\mathbf{s}_n} = 2.2$$

$$\textbf{Player 2}: \qquad \phi_2 = \bar{\mathbf{x}}'B\bar{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \\ \frac{2}{5} \end{bmatrix} = \frac{1}{\mathbf{x}'\mathbf{s}_m} = 2.0.$$

From the two versions of the bimatrix game discussed in this chapter (minimization of expected loss and maximization of expected gain) it is clear that different mixed strategies and expected values of the games are obtained using the same pair of matrices.

Bibliographical References

Headly, G. (1962). *Linear Programming*, Addison-Wesley Publishing Company, Reading, MA.

Lemke, C.E. and J. T. Howson (1964). *Equilibrium Points of Bimatrix Games*, J. Soc. Indust. Appl. Math., 12,413-423

Lemke, C.E. (1968). *On Complementary Pivot Theory*, In G.B. Dantzig and A.F. Veinott Jr, editors, Mathematics of Decision Sciences, Part I, American Mathematical Society, Providence, RI.

Nash, J. (1951). *Non-cooperative Games*, Annals of Math., 54,286:295.