European Journal of Operational Research 129 (2001) 414–433

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

Theory and Methodology

Firms' R&D decisions under incomplete information

V.A. Grishagin [a], Ya.D. Sergeyev [a,b,*], D.B. Silipo [c]

[a] *Software Department, University of Nizhi Novgorod, pr. Gagarina, 23, Nizhi Novgorod, Russian Federation*
[b] *ISI-CNR, c/o University of Calabria, 87036 Rende (CS), Italy*
[c] *Dipartimento di Economia Politica, Università della Calabria, Cubo 19A, 87036 Arcavacata di Rende (CS), Italy*

Received 29 September 1998; accepted 7 September 1999

Abstract

The paper considers a patent race in which firms do not know their relative positions. In this setting, firms that start in the same position proceed at the highest possible speed; and if one firm has an initial advantage it preempts the rival, but at the cost of dissipating a significant part of its monopoly rent. So the paper shows that incomplete information in a patent race leads to rent dissipation. The latter is higher, the higher the value of the prize and the lower the cost of R&D. Thus, for innovations that provide relatively high profits the time to discovery is shortened, but the social losses are likely to be high, due to duplication of effort.

Keywords: Patent races; Incomplete information; Rent dissipation

1. Introduction

Very often the R&D activity that leads to the production of knowledge assumes the characteristics of a race between competing firms. If there is a perfect patent system, the winner takes all and the losers get nothing. On the other hand, if patent protection is imperfect, losers too may benefit from the innovation.

Among the authors who have analyzed models of patent race are Loury (1979), Dasgupta and Stiglitz (1980), Reinganum (1981), Fudenberg et al. (1983), Harris and Vickers (1985, 1987), Beath et al. (1989) and Nti (1997). All assume that firms interact strategically and posit winner-takes-all games.

Although they differ in the characterization of the R&D game (Reinganum (1981) assumes that each firm chooses a time path of R&D expenditure at the outset, whereas Fudenberg et al. (1983) and Harris and Vickers (1985,1987) assume firms revise their decisions according to their relative position in the race), all these models posit that each firm knows its relative position in the race, in terms of acquired knowledge.

Looking at the real world, however, it is very hard to maintain such an assumption. In competitive R&D markets, research programs are conducted secretly, and competitors know very little about the research

* Corresponding author. Tel.: +39-0984-839047; fax: +39-0984-839054.
E-mail address: yaro@si.deis.unical.it (Ya.D. Sergeyev).

PII: S 0 3 7 7 - 2 2 1 7 (9 9) 0 0 4 2 7 - 0

progress of rivals until someone gets the patent. On the other hand, the imperfect information condition in which R&D activity takes place motivates many actions, ranging from industrial espionage to bluffing (overstating their successes in order to induce competitors to drop out of the race).

The aim of the present paper is to analyze a duopoly R&D game when firms do not know their relative positions. More precisely, in a framework with no uncertainty in R&D activity, we consider a duopoly model of patent race, in which the two firms compete for a prize of known value. Although they know their starting positions, they cannot monitor their rival's progress.

In this framework we compute the Nash equilibria of the patent race in relation to the entire parametric space of the game and the initial positions of the firms.

The main conclusion is that both firms will engage in R&D if they are in the same position at the outset. In this case the firms dissipate the rent arising from the patent in the attempt to win the race. On the other hand, if they start in different positions only one firm engages in R&D, although the winner of the race dissipates a significant part of the monopoly rent in order to keep its rival from entering. Thus, the main implication is that in patent races incomplete information leads to rent dissipation.

By contrast, in the models cited if one firm gets a lead on its rival the latter drops out; the race turns out into a monopoly.

On the other hand, one feature of the above models is their inability to explain simultaneous discovery. Even "when firms begin with equal experience there is a burst of R&D followed by the eventual emergence of a monopolist" (Fudenberg et al., 1983, p. 15).

As a matter of fact, in many circumstances several firms make the discovery simultaneously (see Jewkes et al., 1969), as a result of research conducted in parallel and pursuing the same end.

Simultaneous discovery arises in a natural way when firms involved in a deterministic patent race have incomplete information about the position of their rivals. This is due to the fact that competitive firms conducting R&D activity in secret pursue research programs right up to the end of the race, so that if they pursue the same aim, firms that start in the same position get to the end discovery at the same time.

The paper is organized as follows. Section 2 presents the main features of the model, Section 3 computes the Nash equilibrium of the symmetric game, and Section 4 extends the analysis to an initial asymmetric position. Section 5 summarizes the main results.

2. The model

We consider a model in which two firms compete in a multistage patent race for the acquisition of a prize of positive value, V, common to both firms. Like Fudenberg et al. (1983), we assume that the competition is staged in discrete time $t = 0, 1, \ldots,$ and the discovery occurs with certainty when a given number of "units of knowledge", N, are accumulated. The patent is awarded to the firm that first achieves level N. If both firms achieve the discovery simultaneously, the prize goes to the firm with the highest level of knowledge. If they tie, they have equal probability of getting the prize.

Again like Fudenberg et al. (1983), we assume that the R&D process is deterministic. [1] That is, there is a deterministic relationship between the amount of expenditure on the project and the

[1] Of course, the deterministic model of patent race is a simplification of reality. In the real world, the degree of uncertainty related to R&D activity goes from true uncertainty, which characterizes basic research and fundamental inventions and innovations, to very little uncertainty, which is the case of modification of the existing product or process, product differentiation and minor technical improvements (see Freeman and Soete, 1997). One example of the kind of races we are considering here is the race between United States and Germany during the second world war to produce the atomic bomb. Many other examples involve the use of computers through productive processes or in products in many industries. However, the implications of a stochastic R&D process will be discussed in Section 5.

knowledge accumulated. [2] However, Fudenberg et al. (1983) assume that firms know their relative position, so that in every stage each firm can decide whether to continue or to drop out, taking account of the other firm's position.

Unlike these authors, we assume that research programs are secret. Although the firms know their relative position at the beginning of the race, [3] they are not able to monitor the progress of the rival until the game is over.

Let us describe the R&D game. At time $t = 1$, both firms know their relative position in the race and decide their effort levels for each stage of the game. They then accumulate independently units of knowledge, and the game ends when one firm's accumulated knowledge equals or exceeds a given amount, N (the discovery is supposed to occur with certainty at this moment).

Different levels of effort provide different "amounts of knowledge" and have different costs (R&D expenditure). To simplify the analysis, we assume that in each period firms can exert zero-effort, which provides no knowledge and costs nothing, $c_0 = 0$; low level, which generates one "unit of knowledge" and costs $c_1 > 0$; and high-effort, which provides two units of knowledge and costs $c_2 > c_1$. Following Fudenberg et al. (1983) we assume that

$$c_2 > 2c_1. \tag{2.1}$$

A strategy S is a sequence of efforts ("costs") at every stage t, $1 \leqslant t \leqslant T$, where T is the time when the firm terminates its participation in the race.

On the foregoing our assumptions, an alternative way to describe a strategy S is as a sequence of knowledge units α_t obtained at each stage t, $1 \leqslant t \leqslant T$, i.e., $S = (\alpha_1, \alpha_2, \ldots, \alpha_T)$, where $\alpha_t = \{0, 1, 2\}$ depending on whether the firm exerts zero-, low- or high-effort at each stage t.

Let us consider the possible strategies in more details. Among them is "doing nothing", i.e., not participating in the race. This corresponds to the firm's decision to make zero-effort at every stage. Denote this strategy as

$$S_- = (0). \tag{2.2}$$

In this case the firm's payoff does not depend on the choices of the rival; it is nil, since the firm spends nothing but also earns nothing. [4]

Any strategy $S = (\alpha_1, \alpha_2, \ldots, \alpha_T)$ that leads to the firm's completing the race at time T

$$p(S) = \sum_{i=1}^{T} \alpha_i \tag{2.3}$$

denotes the accumulated knowledge of the firm playing the strategy, and

$$d(S) = \sum_{i=1}^{T} c_{\alpha_i} \tag{2.4}$$

denotes the cost.

[2] The discrete nature of the relationship between money and knowledge embodies the assumption that technological progress takes place through jumps in the knowledge, more than by a continuous relationship between the two variables.

[3] There are many ways in which firms may determine their initial relative position. One is the quality of the assets and of the products of the firms, another may be the past performance of the firms in the field. Note, however, that the qualitative results of the paper continue to hold even if firms do not know their initial positions and make decisions on the basis of conjectures about the initial position of the rival.

[4] This corresponds to the assumption that both firms are new entrants in the product market. This assumption does not hold if one firm is already established in the market.

Any strategy S such that $p(S) < N$ (non-finishing strategy) leads to the firm's dropping out of the race. This strategy provides to the firm a payoff $-d(S)$ independent of the actions of the rival.

If $T > 1$, i.e., $S \neq S\_$, then $-d(S\_) < 0$ and it follows that $S\_$ strictly dominates S. In other words, once started it is worthwhile to complete the race. As a consequence, we can eliminate from the analysis all the non-finishing strategies except $S\_$.

Thus, we consider "finishing" strategies that reach the given knowledge level N. Note that for any finishing strategy S we have

$$N \leqslant p(S) \leqslant N + 1. \tag{2.5}$$

Note further that the payoff of each firm depends on its rival's strategy. So if firm i, $i = a, b$, pursues strategy S and its rival pursues S^* (both finishing) the payoff of player i is

$$U_i(S,S^*) = \begin{cases} V - d(S) \;\ldots\; \text{if} \;\ldots\; T(S) < T(S^*) \;\ldots\; \text{or} \;\ldots\; (T(S) = T(S^*) \;\ldots\; \text{and} \;\ldots\; p(S) > p(S^*)), \\ V/2 - d(S) \ldots\ldots\ldots \text{if} \;\ldots\; T(S) = T(S^*) \;\ldots\; \text{and} \;\ldots\; p(S) = p(S^*), \\ -d(S) \;\ldots\; \text{if} \;\ldots\; T(S) > T(S^*) \;\ldots\; \text{or} \;\ldots\; (T(S) = T(S^*) \;\ldots\; \text{and} \;\ldots\; p(S) < p(S^*)). \end{cases}$$

We can rewrite the expressions for the accumulated knowledge $p(S)$ and the expenditure $d(S)$ as

$$p(S) = q_1 + 2q_2, \tag{2.6}$$

$$d(S) = q_1 c_1 + q_2 c_2, \tag{2.7}$$

where q_1 indicates the number of times that firm i exerts low-effort and q_2 the number that it exerts high-effort under strategy S. It is evident that the number of steps for S

$$T(S) = q_1 + q_2. \tag{2.8}$$

Several strategies that achieve $p(S)$ in the same period $T(S)$ can exist. They differ in the order of exerting the same levels of effort, but, because there is imperfect information between players, [5] q_1 and q_2 coincide for all the strategies. As an example for the case $N = 4$, we can indicate the strategies $S' = (1, 1, 2)$, $S'' = (1, 2, 1)$, $S''' = (2, 1, 1)$. Formally they are different but all lead to the same final result.

Thus, each finishing strategy is determined by two parameters q_1 and q_2 only, and we can restrict our analysis to strategies that differ with respect to these two parameters. So without loss of generality we consider the strategy applying first all q_1 (low-effort) and subsequently the remaining q_2 steps with maximum effort as a representative of the entire group (S' in the above example). We denote this strategy as

$$S = (q_1, q_2). \tag{2.9}$$

Let us define the payoff $U_i(S, S^*)$ of the firm i adopting $S = (q_1, q_2)$ while the rival pursues $S^* = (q_1^*, q_2^*)$ as

$$U_i(S,S^*) = \begin{cases} V - q_1 c_1 - \; q_2 c_2 \quad \text{if } q_1 + q_2 < q_1^* + q_2^* \text{ or} \\ \quad (q_1 + q_2 = q_1^* + q_2^* \text{ and } q_1 + 2q_2 > q_1^* + 2q_2^*), \\ V/2 - q_1 c_1 - \; q_2 c_2 \quad \text{if } q_1 = q_1^* \text{ and } q_2 = q_2^*, \\ -q_1 c_1 - q_2 c_2 \quad \text{if } q_1 + q_2 > q_1^* + q_2^* \text{ or} \\ \quad (q_1 + q_2 = q_1^* + q_2^* \text{ and } q_1 + 2q_2 < q_1^* + 2q_2^*). \end{cases} \tag{2.10}$$

[5] Of course, this assumption is not any more valid if firms know their positions in the race. In this case, the order of play is crucial to their subsequent positions and decisions in the race. For the analysis of this case, see Fudenberg et al. (1983) and Harris and Vickers (1985).

In this setting, we next consider the conditions for the existence of equilibrium.

A Nash equilibrium for this game consists of a strategy S_i^* for each firm such that $U(S_i^*, S_j^*) \geqslant U(S_i, S_j^*)$ for each firm and for every $S_i \neq S_i^*,\ i, j = a, b$.

To find the possible Nash equilibrium, we first eliminate dominated strategies and then determine the conditions for the existence of Nash equilibrium with respect to the parametric space of the model (i.e., V, N, c_1 and c_2) and the relative position of the firms at the start of the race.

To this end, we construct all finishing strategies and distribute them in decreasing order with respect to q_1, indicating the corresponding number of steps T to arrive at the finish line. That is

$$\begin{aligned}
&\underbrace{1\cdots1}_{N} &&= S(N,0), && T = N,\\
&\underbrace{1\cdots1}_{N-1}2 &&= S(N-1,1), && T = N,\\
&\underbrace{1\cdots1}_{N-2}2 &&= S(N-2,1), && T = N-1,\\
&\underbrace{1\cdots1}_{N-3}22 &&= S(N-3,2), && T = N-1,\\
& &&\vdots && \\
&\underbrace{1\cdots1}_{N-2k}\underbrace{2\cdots2}_{k} &&= S(N-2k,k), && T = N-k,\\
&\underbrace{1\cdots1}_{N-2k-1}\ \underbrace{2\cdots2}_{k+1} &&= S(N-2k-1,k+1), && T = N-k,\\
& &&\vdots && \\
&\underbrace{2\cdots2}_{k_N} &&= S(0,k_N), && T = k_N,
\end{aligned}$$

where

$$k_N = \begin{cases} N/2 & \text{for } N \text{ even},\\ (N+1)/2 & \text{for } N \text{ odd}. \end{cases}$$

So, we have $N+1$ finishing strategies, which can be summarized by the k_N pairs corresponding to the same number of steps T. (If N is even, the strategy $S(0, k_N)$ will be unique). In the pair for $T = N-k$ the first strategy $S(N-2k, k)$ reaches the discovery strictly and the strategy $S(N-2k-1, k+1)$ exceeds the required amount of knowledge N.

Proposition 1. *Any finishing strategy $S = (q_1, q_2)$ exceeding the final level of knowledge N except the strategy $S(0, k_N) = \underbrace{2\cdots2}_{k_N}$ for N odd is dominated.*

Proof. Consider a finishing strategy $S = (q_1, q_2)$ such that $p(S) = N+1$ and $q_1 \geqslant 1$. For this strategy it is always possible to match the strategy $S' = (q_1', q_2')$ with $q_1' = q_1 - 1,\ \ q_2' = q_2$. Note that $T(S') = T(S) - 1,\ p(S') = N$ (this strategy reaches the discovery strictly, without exceeding).

Compare the payoffs of the strategies S and S' to firm i, $i = a, b$, for all possible strategies S^* of the rival.

If $T^* = T(S^*) > T(S)$ or $T^* = T(S)$ and $p(S^*) = N$ then

$$U_i(S, S^*) = V - q_1c_1 - q_2c_2, \tag{2.11}$$

$$U_i(S', S^*) = V - (q_1 - 1)c_1 - q_2c_2, \tag{2.12}$$

whence

$$U_i(S, S^*) < U_i(S', S^*), \quad i = a, b. \tag{2.13}$$

In the case $S^* = S$ we have

$$U_i(S, S^*) = V/2 - q_1c_1 - q_2c_2$$

and comparing the last equality with (2.12), the inequality (2.13) follows.

If $T^* < T(S)$ the payoff

$$U_i(S, S^*) = -q_1c_1 - q_2c_2$$

but the payoff

$$U_i(S', S^*) = V/2 - (q_1 - 1)c_1 - q_2c_2$$

in the situation $S^* = S'$, or

$$U_i(S', S^*) = -(q_1 - 1)c_1 - q_2c_2$$

in the remaining cases, and the relation (2.13) is again fulfilled. □

Thus, the proposition shows that pursuing a strategy that brings about a level of knowledge exceeding required knowledge N is not optimal. As a consequence we can build up the matrix of the game from the set of strategies $\mathscr{T}$ comprising the strategy $S\_$, all the strategies $S(N - 2k, k)$ for $0 \leqslant k \leqslant k_N$, if N is even or for $0 \leqslant k \leqslant k_N - 1$, if N is odd, and the strategy $S(0, k_N)$ at N odd.

Let us consider all the strategies of the set $\mathscr{T}$ in an "increasing" order in accordance with the following rule. A finishing strategy $S(q_1, q_2) \in \mathscr{T}$ is supposed to be "worse" than a finishing strategy $S' = (q'_1, q'_2)$ if $q_2 < q'_2$.

Eliminating all dominated strategies from the initial full set, we get the reduced matrix of the game by the strategies ordered as follows:

$$S\_ < S(N, 0) < S(N - 2, 1) < \cdots < S(N - 2k, k) < \cdots < S(2, k_N - 1) < S(0, k_N) \tag{2.14}$$

if N is even, and

$$S\_ < S(N, 0) < S(N - 2, 1) < \cdots < S(N - 2k, k) < \cdots < S(1, k_N - 1) < S(0, k_N). \tag{2.15}$$

If N is odd.

For any strategy $S(q_1, q_2) \in \mathscr{T}$ such that $S \neq S\_$ and $S \neq S(0, k_N)$ for N odd we have

$$q_1 = N - 2q_2, \quad p(S) = N, \quad T(S) = N - q_2$$

and, therefore, for any such strategies $S(q_1, q_2)$ and $S' = (q'_1, q'_2)$,

$$S < S' \text{ if and only if } q_2 < q'_2.$$

In the case of N odd and $S = S(0, k_N)$,

$$p(S) = N + 1, \quad T(S) = k_N = (N + 1)/2.$$

Note that under N odd there is a strategy $S(1, k_N - 1)$ for which $T(S) = k_N$ as well, but $p(S) = N$.

Taking into account the above relations we can write (2.14) and (2.15) in the following unified and shorter notation:

$$S_{-} < S_0 < S_1 < \cdots < S_k < \cdots < S_{k_N}, \tag{2.16}$$

where $S_k = S(N-2k, k),\ k = 0,\ k \leqslant k_N$ for N even or $k \leqslant k_N - 1$ for N odd, and $S_{k_N} = S(0, k_N)$ for N odd.

Moreover, we can represent the payoffs (2.10) in a simpler form. First, if $S = S_{-}$, then $U_i(S, S^*) = 0$ for any rival strategy S^*.

In the case $S \neq S_{-}$

$$U_i(S,S^*) = \begin{cases} V - q_1c_1 - q_2c_2 & \text{if } S^* < S, \\ V/2 - q_1c_1 - q_2c_2 & \text{if } S^* = S, \\ -q_1c_1 - q_2c_2 & \text{if } S^* > S. \end{cases} \tag{2.17}$$

When N is even or when N is odd and $S \neq S_{k_N}$, we can write the payoff as

$$U_i(S,S^*) = \begin{cases} V - (N-2q_2)c_1 - q_2c_2 & \text{if } q_2^* < q_2, \\ V/2 - (N-2q_2)c_1 - q_2c_2 & \text{if } q_2^* = q_2, \\ -(N-2q_2)c_1 - q_2c_2 & \text{if } q_2^* > q_2 \end{cases} \tag{2.18}$$

and for N odd and $S = S_{k_N}$ we have

$$U_i(S,S^*) = \begin{cases} V - (N+1)c_2/2 & \text{if } q_2^* < q_2, \\ V/2 - (N+1)c_2/2 & \text{if } q_2^* = q_2. \end{cases} \tag{2.19}$$

Thus, we have defined the possible strategies of the game (they are enumerated in (2.16)) and the payoffs for all possible pairs of strategies.

3. The Nash equilibria of the symmetric game

The aim of the present section is to establish the conditions under which the Nash equilibrium exists, assuming that the firms start the race in the same position. We consider the case in which firms start in different positions in the following section.

Let us first prove two useful lemmas. The first establishes that provided one firm is the winner of the race it is profitable for it to make the lowest effort necessary to achieve the discovery. In formal terms, we have

Lemma 1. *If* $k_N = m > k = 0$, *then*

$$U_i(S_k, S^*) > U_i(S_m, S^*) \tag{3.1}$$

for any $S^* < S_k$ *and* $i = a, b$.

Proof. Note that $m > k$ implies $S_m > S_k$. Since $c_2 > 2c_1$ then

$$(m-k)c_2 > 2(m-k)c_1,$$

$$2kc_1 - kc_2 > 2mc_1 - mc_2,$$

$$V - Nc_1 + 2kc_1 - kc_2 > V - Nc_1 + 2mc_1 - mc_2,$$

$$V - (N-2k)c_1 - kc_2 > V - (N-2m)c_1 - mc_2,$$

i.e., (3.1) is fulfilled for the strategies $S_k = S(N-2k, k)$ and $S_m = S(N-2m, m)$.

If N is odd and $m = k_N$ then $S_{k_N} = S(0, k_N)$ and

$$U_i(S_{k_N}, S^*) = V - k_N c_2, \quad S^* < S_{k_N}.$$

Since $c_2 - c_1 > 0$, then

$$V + c_2 - c_1 - k_N c_2 > V - k_N c_2,$$
$$V - c_1 - (k_N - 1)c_2 > V - k_N c_2,$$
$$U_i(S_{k_N-1}, S^*) > U_i(S_{k_N}, S^*), \quad S^* < S_{k_N},$$

i.e., (3.1) holds for $m = k_N$ as well. □

The second lemma states that if the strategy of doing nothing dominates one finishing strategy, then it dominates all finishing strategies.

Lemma 2. *If the strategy* $S_{\_}$ *dominates a strategy* $S_k,\ k \geqslant 0$, *then* $S_{\_}$ *dominates all the strategies* $S_m,\ m > k$.

Proof. First recall that $U_i(S_{\_}, S^*) = 0$ for any S^*.

For a strategy $S_q,\ q \geqslant 0$,

$$\max_S U_i(S_q, S) = U_i(S_q, S_{\_}) \tag{3.2}$$

since

$$U_i(S_q, S) = U_i(S_q, S_{\_}), \quad S < S_q, \tag{3.3}$$

$$U_i(S_q, S) = U_i(S_q, S_{\_}) - V/2, \quad S = S_q, \tag{3.4}$$

$$U_i(S_q, S) = U_i(S_q, S_{\_}) - V, \quad S > S_q. \tag{3.5}$$

If $S_{\_}$ dominates S_k, then $U_i(S_k, S_{\_}) \leqslant 0$. But, from (3.1), for $m > k$,

$$U_i(S_m, S_{\_}) < U_i(S_k, S_{\_}) \leqslant 0$$

and from (3.2) we have $U_i(S_m, S) < 0$ for any S. □

With the foregoing lemmas, we can prove the following theorem.

Theorem 1. *In the game under consideration there exist the following types of Nash equilibria in pure strategies*:

(i) *if*

$$V \leqslant Nc_1, \tag{3.6}$$

then the pair $(S_{\_}, S_{\_})$ *implements the unique Nash equilibrium*;

(ii) *in the cases*

$$Nc_1 < V < \min\{2Nc_1, (N-2)c_1 + c_2\}, \quad N \geqslant 2, \tag{3.7}$$

or

$$c_1 < V < 2c_1, \quad N = 1, \tag{3.8}$$

the matrix of the game contains two (pure-strategy) Nash equilibria, namely, (S_-, S_0) *and* (S_0, S_-); *and one mixed strategy Nash equilibrium on the two pure-strategy equilibria.*

(iii) *Under conditions*

$$c_2 \geqslant (N+2)c_1, \quad N \geqslant 2, \tag{3.9}$$

$$2Nc_1 \leqslant V \leqslant 2(c_2 - 2c_1), \quad N \geqslant 2, \tag{3.10}$$

or

$$2c_1 \leqslant V \leqslant 2c_2 - 2c_1, \quad N = 1, \tag{3.11}$$

there is the unique Nash equilibrium (S_0, S_0);

(iv) *the unique Nash equilibrium* $(S_{k_N},\ S_{k_N})$ *is also guaranteed by the inequality*

$$V \geqslant 2k_N c_2. \tag{3.12}$$

Proof. See Appendix A.

The theoretical results of the theorem can be interpreted as follows. The effort made by firms starting in the same position depends on the value of the prize relative to its cost. First, firms engage in R&D only if the value is greater than Nc_1, i.e., the cost of achieving the discovery exerting low-effort in each period. Moreover, when the value of prize is low (within the double inequalities (3.7) or (3.8)), there are three Nash equilibria: two pure-strategy ones ((S,S_0) and (S_0,S)), and one mixed-strategy combining the two pure strategies. However, since the payoff of the firms differs in the pure strategies, if there is no coordination both firms want to do R&D and both make losses. So the only plausible Nash equilibrium for the relatively low-value prize is the mixed strategy equilibrium that makes the firms indifferent between the two pure-strategy equilibria.

As the prize becomes more attractive relative to the cost (condition (3.10)), or the cost c_2 is significantly greater than c_1 (condition (3.9)), the optimal behavior for both firms is to move to the finish line at low-effort.

Finally, if the prize exceeds the value $2k_N c_2$, the optimal strategy is to proceed to the goal at the highest effort level only.

However, Theorem 1 does not describe all the possible outcomes of the game. There are some circumstances in which there are no Nash equilibria in pure strategies but only mixed-strategy ones. These circumstances are established in Theorem 2 below.

In the proof of the theorem we will make use of the following sequence.

Let us define for $N \geqslant 1$ the sequences

$$a_1^N, a_2^N, \ldots, a_k^N, \ldots, a_{k_N+1}^N \tag{3.13}$$

such that

$$a_1^1 = 2c_2 - 2c_1, \quad a_2^1 = 2c_2 \qquad \text{if } N = 1, \tag{3.14}$$

$$a_1^2 = \max\{2c_2 - 4c_1, c_2\}, \quad a_2^2 = 2c_2 \qquad \text{if } N = 2, \tag{3.15}$$

$$a_1^N = \max\{2c_2 - 4c_1, (N-2)c_1 + c_2\}, \quad a_2^N = k_N c_2, \quad a_3^N = 2k_N c_2 \qquad \text{if } 3 \leqslant N \leqslant 4, \tag{3.16}$$

and, in the general case $N \geqslant 5$,

$$a_1^N = \max\{2c_2 - 4c_1, (N-2)c_1 + c_2\}, \tag{3.17}$$

$$a_k^N = (N-2k)c_1 + kc_2, \quad 2 \leqslant k \leqslant k_N - 1, \tag{3.18}$$

$$a_{k_N}^N = k_N c_2, \tag{3.19}$$

$$a_{k_N+1}^N = 2k_N c_2. \tag{3.20}$$

Lemma 3. *The sequence (3.13) is strictly increasing.*

Proof. For $N = 1$ and $N = 2$ the assertion of the lemma is obviously fulfilled.

Let $N \geqslant 3$. Then, for $k > 1$, as well as for the situation $k = 1$ and $a_1^N = (N-2)c_1 + c_2$, the proof of $a_k^N < a_{k+1}^N$ can be based on the technique used in Lemma 1; because of its simplicity we do not reproduce it here.

There remains the case $a_1^N = 2c_2 - 4c_1$. If $3 \leqslant N \leqslant 4$, then $a_2^N = 2c_2 > a_1^N$.

Finally, for $N = 5$

$$a_2^N = (N-4)c_1 + 2c_2 = Nc_1 + 2c_2 - 4c_1 > a_1^N. \qquad \square$$

Theorem 2. *Assume that*

$$a_k^N < V < a_{k+1}^N, \quad 1 \leqslant k \leqslant k_N. \tag{3.21}$$

Then, the game has a mixed-strategy Nash equilibrium combining the pure strategies $S_{\_}$, $S_0, \ldots, S_k$.

Proof. See Appendix B.

Thus, for intermediate values of V given by condition (3.21), there exist only mixed-strategy Nash equilibra, in which both firms randomize between the first k pure strategies.

Now we are able to provide the full classification of Nash equilibria in relation to the values of the parameters of the game: i.e., V, N, c_1, c_2 when the firms start in the same position in the race. This classification is given with respect to an increasing order of the value of the prize, V.

There are two cases as regards the R&D technology:

$$(1)\ 2c_1 < c_2 < (N+2)c_1 \tag{3.22}$$

and

$$(2)\ c_2 \geqslant (N+2)c_1. \tag{3.23}$$

Consider first (3.22). It follows from the foregoing that if

(1.1) $V \leqslant Nc_1$, we have unique Nash equilibrium $(S_{\_}, S_{\_})$ in pure strategies. For

(1.2) $Nc_1 < V \leqslant (N-2)c_1 + c_2$ there exist two Nash equilibria in pure strategies $(S_{\_}, S_0)$ and $(S_0, S_{\_})$ and one mixed-strategy equilibrium. Up to

(1.3) $V = 2k_N c_2$ only the mixed strategies are implemented. If

(1.4)

$$(N-2k)c_1 + kc_2 < V \leqslant (N-2(k+1))c_1 + (k+1)c_2, \tag{3.24}$$

$1 \leqslant k \leqslant k_N$, then we have the equilibrium state in mixed strategies over pure strategies $(S_{\_}, S_0, S_1, \ldots, S_k)$. Finally, for

(1.5)

$$V > 2k_N c_2, \tag{3.25}$$

we again have a unique Nash equilibrium in pure strategies: (S_{k_N}, S_{k_N}).

In case (3.23) and

(2.1) $V \leqslant Nc_1$, there exists a unique Nash equilibrium (S, S). But under the condition

(2.2) $Nc_1 < V \leqslant 2Nc_1$ two pure-strategy equilibria $(S_{\_}, S_0)$ and $(S_0, S_{\_})$ and one mixed-strategy one are implemented; However, if

(2.3) $2Nc_1 < V \leqslant 2c_2 - 4c_1$ we again have the single Nash equilibrium (S_0, S_0). When

(2.4) $2c_2 - 4c_1 < V \leqslant (N-4)c_1 + 2c_2$, there arises the mixed equilibrium state over the pure strategies $(S_{\_}, S_0, S_1)$. Finally, for the conditions

(2.5) (3.24) and (3.25) we have the same results as in the case (3.22).

This classification has at least two significant implications: (1) competition in a patent race is fiercer as the value of the prize increases relative to the cost of discovery, in that more aggressive strategies are required to win; (2) the higher is c_2 relative to c_1, the greater the incentive for the firms to play pure strategy instead of mixed-strategy Nash equilibria.

It is intuitive that whatever strategies are adopted by symmetric firms, their expected profit is nil if there are no constraints on their speed. [6] To prove this, assume instead that at a Nash equilibrium one of the firms does get a positive profit. It follows that the rival, by increasing the level of R&D expenditure, can get to the discovery earlier without making negative profit; this in turn would lead the first firm to increase its own R&D effort. Proceeding thus, we conclude that the firms increase expenditure up to the point where their expected profit is nil. The only conclusion is that in the effort to win the race firms starting in symmetric positions dissipate the rent arising from the patent.

In Section 4 we show that the rent dissipation result holds even when starting positions are different.

4. The asymmetric game

We now consider a situation in which one firm has an advantage over the other firm at the beginning of the race, due to such factors as differences in size, market position or assets, and this advantage is common knowledge to both.

Let us assume at the outset that firm i is $k \geqslant 1$ steps ahead of firm j, $i, j = a, b$. That is, firm i needs to accumulate only $N - k > 0$ units of knowledge when j must accumulate N units to get to the discovery. The following theorem then holds.

Theorem 3. *If firm i is $k \geqslant 1$ steps ahead of firm j, there always exists a strategy allowing i to win the race, and therefore forcing j not to play*, $i, j = a, b$.

[6] For example, for the case $V \geqslant 2k_N c_2$, the firms make positive profits at a Nash equilibrium. However, this result depends on the constraint on available options to the firms they can play only $\{0, 1, 2\}$. Without this constraint the rent dissipation would be total.

Proof. Assume the firms are at $t = 1$. Further, without loss of generality, assume that at the beginning of the race firm a is k-steps closer to the discovery than firm b. Thus, firm a wins the game if it accumulates an amount of knowledge equal to $N - k$, that is

$$q_1 + 2q_2 = N - k$$

and it arrives before firm b, i.e.,

$$q_1 + q_2 < q_1^* + q_2^*,$$

where, of course,

$$q_1^* + 2q_2^* = N.$$

It follows from the above inequalities that firm a wins if

$$q_2 > q_2^* - k. \tag{4.1}$$

Firm b wants to catch up with firm a and will run at its top possible speed insofar as it does not make losses; that is, it pursues strategy (q_1^*, q_2^*), where

$$q_2^* = \max\{q_2^* :\ V - c_1 q_1^* - c_2 q_2^* \geqslant 0,\ q_1^* + 2q_2^* = N\}. \tag{4.2}$$

Let us first consider the case when V is high enough to allow to firm b to run at top speed, i.e.,

$$q_2^* = I(N/2) \quad \text{and} \quad V - c_1 q_1^* - c_2 q_2^* \geqslant 0,$$

where $I(r)$ is the integer part of r. Thus, it follows from (4.1) and $k \geqslant 1$ that there exists a strategy (q_1, q_2) for firm a $(0 \leqslant q_2 \leqslant I(N/2))$ such that

$$q_2 > I(N/2) - k,$$

and a wins the game because b cannot catch up and therefore will not participate.

When the value V is not high enough to allow to firm b to run at top speed, its speed is bounded by the inequality

$$V - c_1 q_1^* - c_2 q_2^* \geqslant 0.$$

Thus, firm a wins again, because given (4.1) and $k \geqslant 1$, it is in a position to make the highest effort enough times that it is uneconomic for b to participate.

So at $t = 1$ firm b drops out and firm a makes the highest effort. □

In order to demonstrate the stability of the solution reached, we show that firms do not deviate from the above equilibrium conditions.

Assume at $t = 1$ that firm a is closer to the discovery than firm b by one step. [7] At $t = 1$ firm b does not participate and firm a makes the highest effort (see Theorem 3).

Therefore, at time $t = 2$ firm a is three steps ahead of firm b, and consider the behavior of the firms at this stage. To show that also in this stage firm a makes the highest effort, let assume that at $t = 2$ firm a changes its strategy to low-effort. In this case, firm b would enter the race at the outset and catch up at time $t = 2$. So, in order to keep firm b from participating, firm a must make the highest effort also in the second period. By induction, assume this is true up to stage $T - 2$, and consider the behavior of the firms at stage

[7] This result holds *a fortiori* if $k > 1$.

$T-1$. Assume firm a plays one at this stage. In this case, firm b would choose to enter at the beginning of the race and make the highest effort in each period, because this strategy guarantees a neck-and-neck (finish). It follows that a firm that has a one-step headstart, cannot deviate from the highest effort level even in the last period. [8]

So when one firm is ahead of its rival, it runs to the finish line at the highest possible speed.

The main implication of the theorem is thus that, with incomplete information, the leader must dissipate all or a significant part of the rent from the discovery in order to win the race. This result contrasts sharply with the case of complete information in patent races (see, for example Harris and Vickers, 1985, 1987; Fudenberg et al., 1983). On the other hand, it is in line with the recent result of Nti (1997). In a symmetric patent race with complete and perfect information, the latter proves that the profits of the firms go to 0 as the number of rivals increase. In this paper we have shown that if there is incomplete information this result is also valid when there are only two firms. [9]

Finally, let us consider the effects of an increase in the value of the prize and in the R&D costs on the speed of R&D when firms are in an asymmetric position.

By conditions (4.1) and (4.2), it is straightforward to conclude that an increase in V determines an increase in the number of times that firm b can make the highest effort. It follows that an increase in the value of the prize intensifies competition, in that it induces the leader to make the greatest effort for a longer period, thus shortening the time to discovery.

On the other hand, an increase of c_2 relative to c_1 has the opposite effect, because it decreases the number of times that it is profitable for both firms to make the top effort.

5. Conclusions

We have considered a deterministic model of a patent race, in which firms know their initial position but are not able to monitor the progress made by their rival. In this framework, we study the nature of the R&D race in relation to the position of the firms and the values of the parameters.

With respect to position, we have proved that firms that start in the same position get the discovery simultaneously, while an initial lead is enough to ensure the patent to the firm with the head start. However, the winner dissipates much or all of the rent from the innovation in the competition to be the first.

Relative to the second aspect, we have shown that competition in the patent race is the more vigorous, the greater the value of the prize. Other things equal, this leads to the conclusion that the higher the value of the prize, the shorter the time to discovery.

Since the model is similar to that of Fudenberg et al. (1983), apart from the latter's assumption that there exists a one-period information lag (in our model the information lag is infinite), it may be interesting to compare results. Fudenberg et al. (1983) show that when firms begin with equal experience the race is characterized by vigorous competition in the early stages, followed by the eventual emergence of a monopolist, and if one firm lags two or more steps behind, the race becomes a monopoly. In a similar framework, Harris and Vickers (1985,1987) also reach the same conclusions. One implication is that an increase in the value of the prize intensifies competition if firms are in similar positions, but does not affect the level of R&D expenditure when a firm lags two or more steps behind.

---

[8] This conclusion is strictly related to the fact that firms are never able to monitor the behavior of the rival during the race and revise their decisions accordingly. See also Footnote 6.

[9] There are, however, some important differences between Nti's paper and ours. The former is settled in a static framework, while ours adopts a dynamic approach. A extensive discussion of many contexts in which rent dissipation takes place is provided by Fudenberg and Tirole (1987).

By contrast, our own results show that with an infinite information lag the leader never proceeds at the monopoly pace, and the firms intensify their efforts as the value of the prize increases, independently of relative position.

However, the most striking result of Fudenberg et al. (1983) is that the impossibility of monitoring current decisions of the rivals in some circumstances may enable the tailing firm to leapfrog its rival. However, this result is no longer valid if we assume that the firms can never monitor the position of the rival. In the latter case, the leader preempts the rival when he has even a small advantage.

Even though it is very difficult to provide evidence of the role of incomplete information in patent races, many authors (e.g., Rothwell, 1992,1994; Dodgson, 1991; Maidique and Zirger, 1985) support the view that, when a firm achieves success with a radical innovation, this is frequently followed by an accumulative series of further successful innovations in the same field. These findings may be explained by drawing on the hypothesis that incomplete information in patent races leads the leader of the race to adopt an "offensive" strategy rather than resting on its laurels and consolidating its established position (see, Freeman and Soete, 1997, chapter 11). On the other hand, were the relative positions of the firms in the race to be known, it would be possible for the leader which is sufficiently ahead to proceed at the monopoly pace (see, Fudenberg et al., 1983).

Although our results seem to support the adage that "nothing succeeds like success", the extension of the analysis to a sequence of innovation is not a matter of replicating equilibria of the single-race game. It may well be the case that leapfrogging can occur along the sequence of possible innovations even in deterministic models of patent races, if the trailing firm can outweigh the losses of the earlier innovations with the gains of the later ones.

A different type of behavior may also arise in a framework of uncertainty over the results of R&D activity. It may well be the case that the firm that is behind has an incentive to enter the race because it has a chance to win. However, these two extensions are beyond the scope of the present paper. Finally, although some results of our paper are in line with previous results on this topic (notably, Nti, 1997; Harris and Vickers, 1985,1987; Fudenberg et al., 1983; Reinganum, 1981), our conclusions suggest that rent dissipation in patent races is a more extensive phenomenon than is generally supposed.

Acknowledgements

We are grateful to Vincenzo Denicolò and three anonymous referees for helpful comments on a previous version of this paper.

Appendix A. Proof of Theorem 1

A.1. Assertion (i)

Because of $V \leqslant Nc_1$, $U_i(S_0, S_\_) = V - Nc_1 \leqslant 0 \quad i = a, b$, and according to (3.4) and (3.5), $U_i(S_0, S) < U_i(S_0, S_\_)$ for all $S > S_\_$, i.e., S dominates S_0 and, therefore, all other strategies S_k, $k_N \geqslant k \geqslant 1$.

Thus, there is the unique Nash equilibrium $(S_\_, S_\_)$.

A.2. Assertion (ii)

Assume that $N \geqslant 2$. It means that there exists a strategy $S_1 = S(N-2, 1)$ for which $U_i(S_1, S^*) \leqslant V - (N-2)c_1 - c_2 < 0$ for any strategy S^* of the rival. Therefore, $S\_$ dominates S_1 and, according to Lemma 2, all the strategies $S_k,\ k > 1$. After eliminating dominated strategies, we can reduce the matrix of the game up to the 2×2 matrix

	S_-	S_0
S_-	0, 0	$0,\ V - Nc_1$
S_0	$V - Nc_1, 0$	$V/2 - Nc_1,\ V/2 - Nc_1$

where the first column represents the possible strategies of firm a and the first row the possible strategies of firm b. So, the left-hand number in a cell contains the payoff of player a, and the right-hand indicates the payoff of player b.

Let us denote as $R^1(S)$ the best response of the ith player, $i = a, b$, to a strategy S of the rival. Then, $R^i(S_-) = S_0$, since $V > Nc_1$, $R^i(S_0) = S_-$, since $V < 2Nc_1$, and we have two Nash equilibria in pure strategies: (S_0, S_-) and (S_-, S_0).

If $N = 1$, then $S_1 = S(0, 2)$ and it is dominated by S, because $V < 2c_1 < c_2$. For $N = 1$, after elimination of S_1 from the matrix we obtain the 2×2 matrix (see above). As $c_1 < V < 2c_1$, the situation in this matrix is the same: two Nash equilibria (S_0, S_-) and (S_-, S_0).

Moreover, there always exist (see Dasgupta and Maskin, 1986) a mixed-strategy Nash equilibrium which makes each firm indifferent between the two pure-strategy equilibria.

A.3. Assertion (iii)

Let us compare the strategy $S_0 = S(N, 0)$ with the nearest strategy S_1. If $N \geqslant 2$, then $S_1 = S(N-2, 1)$. Take an arbitrary strategy S^* of the rival. If $S^* = S_-$, then

$$\begin{aligned} U_i(S_0, S^*) &= V - Nc_1, \\ U_i(S_k, S^*) &= V - (N-2)c_1 - c_2, \end{aligned} \tag{A.1}$$

and due to $c_2 > 2c_1$ we have

$$U_i(S_k, S^*) < U_i(S_0, S^*). \tag{A.2}$$

If $S^* = S_0$, then in accordance with (3.10)

$$\begin{aligned} U_i(S^k, S_0) = V - (N-1)c_1 - c_2 &\leqslant V/2 + c_2 - 2c_1 - (N-2)c_1 - c_2 \\ &= V/2 - Nc_1 = U_i(S_0, S_0). \end{aligned}$$

Now let $S^* = S_k$ In this case

$$\begin{aligned} U_i(S_0, S^*) &= -Nc_1, \\ U_i(S_k, S^*) &= V/2 - (N-2)c_1 - c_2. \end{aligned} \tag{A.3}$$

Again from the right-hand side of (3.10)

$$U_i(S_k, S^*) \leqslant c_2 - 2c_1 - (N-2)c_1 - c_2 = -Nc_1 = u(S_0, S^*).$$

At last, if $S^* > S_k$, then $U_i(S_0, S^*)$ is taken as (A.3) and

$$U_i(S_k, S^*) = -(N-2)c_1 - c_2 = -Nc_1 - (c_2 - 2c_1) < -Nc_1 = U_i(S_0, S^*),$$

i.e., (A.2) is fulfilled.

So, we have proved that strategy S_0 dominates strategy S_1. Therefore, we can delete S_1 from the matrix.

Proceeding by induction on k, assume that for $q = k-1,\ k > 1$, all the strategies $S_m = S(N-2m, m),\ m \leqslant q$, are excluded. This means that the strategy $S_k = S(N-2k, k),\ k > 1$, will be the nearest such that $S_k > S_0$.

Let us choose the traditional way of comparing the payoffs for the strategies S_0 and S_k.

First, notice that $V/2 - k(c_2 - 2c_1) < 0,\ k > 1$, since

$$V/2 \leqslant c_2 - 2c_1 < k(c_2 - 2c_1), \quad k > 1. \tag{A.4}$$

If $S^* = S_0$ then

$$U_i(S_k, S^*) = V - (N-2k)c_1 - kc_2 = V - Nc_1 - k(c_2 - 2c_1) < V/2 - Nc_1 = U_i(S_0, S^*).$$

In the case $S^* = S\_,\ U_i(S_k, S\_) = U_i(S_k, S_0)$, but

$$U_i(S_0, S\_) = V - Nc_1 > U_i(S_0, S_0)$$

and (A.2) holds again.

The next case: $S^* = S_k$. We have (A.3) for $U_i(S_0, S^*)$ and

$$U_i(S_k, S^*) = V/2 - (N-2k)c_1 - kc_2.$$

Taking into account (A.4) the inequality (A.2) holds, as well as in the situation $S^* > S_k$ when (A.3) holds for $U_i(S_0, S^*)$, but

$$U_i(S_k, S^*) = -(N-2k)c_1 - kc_2.$$

Thus, the strategy S_k is also dominated by S_0.

So, all the strategies of type $S_k = S(N-2k, k),\ k = 1$ are dominated, and there remain only the two strategies $S\_ = S(0,0)$ and $S_0 = S(N, 0)$ for N even, or three strategies S_0, S_1 and $S_{k_N} = S(0, k_N)$ for N odd, $N \geqslant 1$. In the last case, we have

$$U_i(S_0, S_0) = V/2 - Nc_1, \quad U_i(S_k, S_0) = V - k_N c_2.$$

But

$$V/2 \leqslant c_2 - 2c_1 \leqslant N(c_2 - 2c_1)/2 < k_N(c_2 - 2c_1), \quad N \geqslant 3, \tag{A.5}$$

i.e.,

$$\begin{aligned} U_i(S_{k_N}, S_0) = V/2 + V/2 - k_N c_2 &< V/2 - k_N c_2 + k_N(c_2 - 2c_1) \\ &= V/2 - 2k_N c_1 = V/2 - Nc_1 - c_1 < V/2 - Nc_1 \\ &= U_i(S_0, S_0). \end{aligned}$$

The domination of $U_i(S_0, S\_)$ over $U_i(S_{k_N}, S\_)$ is given by Lemma 2.

The case $S^* = S_{k_N}$ is the duplication of the variant $S^* = S_0$ because of $U_i(S_0, S_{k_N}) = U_i(S_0, S_0) - V/2$ and $U_i(S_{k_N}, S_{k_N}) = U_i(S_{k_N}, S_0) - V/2$.

The situation $N = 1$ must be considered separately, because in this case (A.5) does not hold.

Compare again $U_i(S_0, S_0). = V/2 - c_1$ and $U_i(S_1, S_0) = V - c_2$. From (3.11) $V/2 < c_2 - c_1$, hence,

$$V - c_2 < V/2 - c_1.$$

The domination of S_0 over S_1 for $S^* = S\_$ is obvious.

So, our initial matrix can be reduced to 2×2 with the strategies $S\_$ and S_0. The domination of S_0 over $S\_$ in this matrix follows immediately from the left-hand sides of the inequalities (3.10) and (3.11), since

$$U_i(S_0, S\_) = V - Nc_1 > U_i(S_0, S_0) = V/2 - Nc_1 \geqslant 0.$$

Thus, the proof of assertion (iii) is complete.

A.4. Assertion (iv)

Consider the strategy S_{k_N} and evaluate its payoffs.

$$U_i(S_{k_N}, S_{k_N}) = V/2 - k_N c_2 \geqslant 0,$$
$$U_i(S_{k_N}, S^*) = V - k_N c_2 > 0, \quad S^* < S_{k_N},$$

from which it follows that S_{k_N} dominates $S\_$.

Delete $S\_$ from the matrix and compare the strategies S_{k_N} and S_0. As $U_i(S_0, S^*) = -Nc_1 < 0$, if $S^* > S_0$. To prove the domination of S_{k_N} over S_0 it is sufficient to show that

$$U_i(S_0, S_0) < U_i(S_{k_N}, S_0).$$

Recalling (3.12) we have

$$V > 2k_N c_2 - 2Nc_1, \quad V/2 - k_N c_2 > -Nc_1, \quad V - k_N c_2 > V/2 - Nc_1.$$

The last inequality proves that S_{k_N} dominates S_0.

Proceeding by induction on $k < k_N - 1$ assume that all the strategies $S_0, \ldots, S_k$, $k \geqslant 0$, have been excluded from the matrix (the strategy $S\_$ is supposed to have been deleted from the initial matrix before the induction).

For the strategy S_{k+1}

$$U_i(S_{k+1}, S^*) = -(N - 2(k+1))c_1 - (k+1)c_2 < 0, \quad S^* > S_{k+1}.$$

Meanwhile, from (3.12)

$$V/2 \geqslant k_N c_2 > k_N c_2 - Nc_1 - (k+1)(c_2 - 2c_1)$$

or, after simple transformations,

$$V - k_N c_2 > V/2 - (N - 2(k+1))c_1 - (k+1)c_2,$$

which implies

$$U_i(S_{k+1}, S_{k+1}) < U_i(S_{k_N}, S_{k+1})$$

and the last inequality completes the proof of assertion (iv) and of the theorem. □

Appendix B. Proof of Theorem 2

As a first step, let us show that any strategy S_m, $m > k$, is dominated by the strategy $S\_$. This becomes clear immediately if we indicate that

$$U_i(S_m, S^*) = V - a_m^N \leqslant 0, \quad S^* < S_m, \tag{B.1}$$

$$U_i(S_m, S_m) = V/2 - a_m^N < 0, \quad \text{(B.2)}$$

$$U_i(S_m, S^*) = -a_m^N < 0, \quad S^* > S_m, \quad \text{(B.3)}$$

$i = a, b$.

Eliminate all the strategies $S_{k+1}, \ldots, S_{k_N}$ from the matrix (if such strategies exist).

Before the continuation of the proof notice that

$$a_1^N > Nc_1. \quad \text{(B.4)}$$

For $N = 1$ and $N = 2$ (B.4) is true because of $c_2 > 2c_1$, and if $N \geqslant 3$, then

$$a_1^N = (N-2)c_1 + c_2 = Nc_1 + c_2 - 2c_1 > Nc_1.$$

Now let us obtain the best responses $R^i(S)$ of player i, $i = 1, 2$, to all possible variants S $\{S_{\_}, S_0, \ldots, S_k\}$ of rival's behavior in the new matrix.

Let the rival of the player i adopt strategy $S_{\_}$. Then,

$$R^i(S_{\_}) = S_0$$

since $U_i(S_0, S_{\_}) > 0$ because of (3.21) and (B.4) and $U_i(S', S_{\_}) < U_i(S_0, S_{\_}),\ S' > S_0$, according to Lemma 1.

Consider a strategy S_q of the rival, $1 \leqslant q \leqslant k-1$, and show that the best response

$$R^i(S_q) = S_{q+1}. \quad \text{(B.5)}$$

Indeed,

$$U_i(S_{q+1}, S_q) = V - a_{q+1}^N = V - a_k^N > 0,$$

i.e.,

$$U_i(S_{q+1}, S_q) > U_i(S_{\_}, S_q).$$

Moreover,

$$U_i(S_m, S_q) = -a_m^N < 0, \quad 0 \leqslant m < q,$$

and

$$U_i(S_m, S_q) < U_i(S_{q+1}, S_q), \quad q+1 < m \leqslant k_N,$$

due to Lemma 1.

Finally, from (3.21) and (3.15)–(3.17) for $N \geqslant 2$,

$$V > a_k^N \geqslant a_1^N \geqslant 2c_2 - 4c_1. \quad \text{(B.6)}$$

So, for $q < k \leqslant k_N$, if N is even, and for $q < k < k_N$, if N is odd, it follows that

$$V/2 > c_2 - 2c_1 = (N - 2(q+1))c_1 + (q+1)c_2 - (N-2q)c_1 - qc_2$$

or

$$V - a_{q+1}^N > V/2 - a_q^N, \quad \text{(B.7)}$$

i.e.,

$$U_i(S_{q+1}, S_q) > U_i(S_q, S_q). \quad \text{(B.8)}$$

If $N = 1$ is odd and $k = k_N,\ q = k_N - 1$, then according to (B.6)

$$V > a^N_{k_N} = k_N c_2 = 2c_2 > 2c_2 - 2c_1, \quad N \geqslant 3$$

(if $N = 1$ then $V > 2c_2 - 2c_1$ follows direct from (3.14)). But then

$$V/2 > c_2 - c_1 = c_2 - (N - 2(k_N - 1))c_1$$

or

$$V - k_N c_2 > V/2 - (N - 2(k_N - 1))c_1 - (k_N - 1)c_2,$$

i.e., (B.8) holds again.

Thus, we have shown that the maximal payoff of the ith player given the rival's strategy $S_q,\ q < k$, is achieved by adopting S_{q+1}, i.e., the best response is described by (B.5).

Consider the last case, when the rival plays S_k. We have for $0 \leqslant m < k$

$$U_i(S_m, S_k) = -(N - 2m)c_1 - mc_2 < 0.$$

From (3.21)

$$V \leqslant a^N_{k+1} = a^N_k + c_2 - 2c_1$$

if $1 < k \leqslant k_N - 1$ at N even or $1 < k < k_N - 1$ at N odd. But

$$c_2 - 2c_1 < 2c_2 - 4c_1 \leqslant a^N_k,$$

i.e., $V < 2a^N_k$, or

$$U_i(S_k, S_k) = V/2 - a^N_k < 0.$$

If $k = 1$, then

$$V \leqslant a^N_2 = (N - 4)c_1 + 2c_2 < (N - 4)c_1 + 2c_2 + Nc_1 = 2[(N - 2)c_1 + c_2],$$

whence

$$U_i(S_k, S_k) = V/2 - (N - 2)c_1 - c_2 < 0.$$

Let $k = k_N - 1$ and $N \geqslant 3$ odd. Then

$$V \leqslant a^N_{k_N} = k_N c_2 < k_N c_2 + 2c_1 + (k_N - 2)c_2 = 2((k_N - 1)c_2 + c_1),$$

whence

$$U_i(S_k, S_k) = V/2 - c_1 - (k_N - 1)c_2 < 0.$$

Consider the last subcase $k = k_N$. Then $V < 2k_N c_2$ and

$$U_i(S_k, S_k) = V/2 - k_N c_2 < 0.$$

It follows that

$$R^i(S_k) = S_-.$$

and the last condition completes the proof. □

References

Beath, J., Katsoulacos, Y., Ulph, D., 1989. Strategic R&D policy. The Economic Journal 99, 74–83.
Dasgupta, P., Maskin, E., 1986. The existence of equilibrium in discontinuous economic games. 1: Theory. Review of Economic Studies 53, 1–26.
Dasgupta, P., Stiglitz, J., 1980. Uncertainty, industrial structure and the speed of R&D. Bell Journal of Economics 90, 266–293.
Dodgson, M., 1991. Technological learning, technology strategy and competitive pressures. British Journal of Management 2 (3).
Freeman, C., Soete, L., 1997. The Economics of Industrial Innovation, third ed. Pinter, London.
Fudenberg, D., Gilbert, R., Stiglitz, J., Tirole, J., 1983. Preemption, leapfrogging and competition in patent races. European Economic Review 22, 3–31.
Fudenberg, D., Tirole, J., 1987. Understanding rent dissipation: on the use of game theory in industrial organization. The American Economic Review 77 (2), 176–183.
Harris, C., Vickers, J., 1985. Perfect equilibrium in a model of a race. Review of Economic Studies LII, 193–209.
Harris, C., Vickers, J., 1987. Racing with uncertainty. Review of Economic Studies LIV, 1–21.
Jewkes, J., Sawers, D., Stillerman, R., 1969. The Sources of Invention. Norton & Company, New York.
Loury, G.C., 1979. Market structure and innovation. Quarterly Journal of Economics 93, 395–410.
Maidique, M.A., Zirger, B.J., 1985. The new product learning cycle. Research Policy 14, 299–313.
Nti, K.O., 1997. Comparative statics of contests and rent-seeking games. International Economic Review 38 (1), 43–59.
Reinganum, J., 1981. Dynamic games of innovation. Journal of Economic Theory 125, 21–41.
Rothwell, R., 1992. Successful industrial innovation: Critical factors for the 1990s. R&D Management 22 (3), 221–239.
Rothwell, R., 1994. Industrial innovation: Success, strategy, trends. In: Dodgson, M., Rothwell, R. (Eds.), Handbook of Industrial Innovation. Elgar, Cheltenham.