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NEW DEVELOPMENTS ON THE USE OF BIVARIATE RODRIGUEZ-BURR III DISTRIBUTION IN RELIABILITY STUDIES

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Abstract

In this paper we study the bivariate Rodriguez-Burr III distribution from a reliability point of view. In particular, we derive various functions used in reliability theory of conditional distributions, viz hazard rate, reversed hazard rate, mean residual life and mean reversed residual life and, using some notions of dependence, their monotonicity is discussed. Finally, some measures of dependence based on the distribution function and on the mean reversed residual life are investigated.

Key words: Conditional Distribution, Reveserd Hazard Rate, TP2, Dependence Measures.

1 Introduction

It is well-known in the literature that the Burr III distribution is the third example of solutions of the differential equation defining the Burr system of distribution (Burr, 1942). This distribution has been widely used in various fields of sciences, in some cases with different parameterizations and under other names. For example, it is called inverse Burr distribution in the actuarial literature (see, e.g., Klugman et al., 1998) and kappa distribution in the metereological literature (Mielke, 1973; Mielke and Johnson, 1973). A generalization of Burr III model, called Dagum distribution, has been successfully used in studies on income and wage distribution as well as in those on wealth distribution (see Dagum, 1977, 1980; Kleiber and Kotz, 2003; Quintano and D'Agostino, 2006; Kleiber, 2007; Domma, 2007). The Burr III distribution has been employed in financial literature, environmental studies, in survival and reliability theory (see, i.e., Sherrick et al. (1996); Lindsay et al. (1996); Gove et al. (2008); Shao (2000); Hose (2005); Al-Dayian (1999); Mokhlis (2005)). Recently, Shao et al. (2008) proposed the use of the so-called extended Burr III distribution in low-flow frequency analysis where the lower tail of a distribution is of interest.

Rodriguez (1980) proposed the extension to the bivariate case of univariate Burrr III distribution and derived the conditional density, conditional moments and correlation index. Since then, papers on the bivariate Rodriguez-Burr III distribution has been rather skimpy comparated with the work that has been carried out, for example, on bivariate Burr XII. Recently, some authors studied various reparameterization and/or special cases of the model proposed by Rodriguez. For example, studying the relation between the functional and personal distribution of income, Dagum (1999) obtained a reparameterization of bivariate Rodriguez-Burr III for modelling the distribution between human capital and wealth. Bismi G. Nadh et al. (2005) provided a general method of generating multivariate Burr distribution extending the differential equation proposed by Burr (1942) to higher dimensions. Solving the corresponding set of partial differential equations, they obtained the bivariate Burr system of distributions of which the type III is a member; moreover, they calculated some functions (such as, for example, survival function and reversed hazard rate) useful in the reliability theory. Yari and Mohammad-Djafari (2008) determined the exact form of the Fisher information matrix of a special case of the Rodriguez-Burr III distribution. Studying some properties and indices of dependence of the bivariate Rodriguez-Burr III distribution, Domma (2009 a) proved that the model can describe also situations of negative dependence. Evidently, this results permits to extend the range of potential application of the bivariate Rodriguez-Burr III distribution in various fields of sciences.

In this paper, we study the bivariate Rodriguez-Burr III distribution of reliability point of view. In particular, using some notions of dependence, we analyse the behaviour of various functions, used in reliability theory, of distributions of a random variable X given $Y = y$, of X given $Y < y$ and of X given $Y > y$. Moveover, we calculate several dependence measures used in reliability theory and survival analysis.

The paper is organized as follows. In *Section 2*, we introduce the model and we describe its main features. Moreover, we prove that the bivariate Rodriguez-Burr III density function is $TP2$ (totally positive of order 2). *Section 3* contains some definitions and background of reliability functions and some notions of dependence. The hazard rate, the reversed hazard rate and other functions of the conditional distributions used in reliability theory and their monotonicity are discussed in *Section 4*. Properties of some measures of dependence are investigated in *Section 5*.

2 The Model

In this section, we briefly introduce the bivariate and the conditional Rodriguez-Burr III distributions and we prove that the bivariate density function is $TP2$.

A random vector (X, Y), with X and Y continuous and non-negative random variables, is Rodriguez - Burr III distributed if its joint distribution function is

$$F_{XY}(x, y; \boldsymbol{\xi}) = \left(1 + \alpha\lambda\gamma x^{-\theta} y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta}\right)^{-\beta}, \tag{1}$$

where $\boldsymbol{\xi} = (\beta, \lambda, \gamma, \delta, \theta, \alpha)$ with $\lambda > 0, \gamma > 0, \delta > 0, \theta > 0, \beta > 0$ and $0 \leq \alpha \leq (\beta + 1)$, with bivariate density function

$$\begin{aligned} f_{XY}(x, y; \boldsymbol{\xi}) &= \beta\lambda\gamma\delta\theta x^{-\theta-1} y^{-\delta-1} \left(1 + \alpha\lambda\gamma x^{-\theta} y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta}\right)^{-\beta-2} \times \\ & \quad \left\{(\beta+1)(1+\alpha\lambda x^{-\theta})(1+\alpha\gamma y^{-\delta}) - \alpha\left(1 + \alpha\lambda\gamma x^{-\theta} y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta}\right)\right\} \end{aligned}$$

(see Rodriguez, 1980 and 1983). It is simple to verify that the marginal distributions of X and Y are Burr III (Burr 1942, Dagum 1977) with distribution function given by $F_X(x; \beta, \lambda, \theta) = \left(1 + \lambda x^{-\theta}\right)^{-\beta}$ and $F_Y(y; \beta, \gamma, \delta) = \left(1 + \gamma y^{-\delta}\right)^{-\beta}$, respectively. Furthermore, the conditional density function and conditional distribution function of X given $Y = y$, respectively, are

$$f_{Y=y}(x|y; \boldsymbol{\xi}) = \lambda\theta x^{-\theta-1}(1 + k_y\lambda x^{-\theta})^{-\beta-2}\left[k_y(\beta+1)(1+\alpha\lambda x^{-\theta}) - \alpha(1 + k_y\lambda x^{-\theta})\right] \quad (2)$$

$$F_{Y=y}(x|y; \boldsymbol{\xi}) = (1 + k_y\lambda x^{-\theta})^{-\beta-1}(1 + \alpha\lambda x^{-\theta}), \quad (3)$$

where $k_y = \frac{(1+\alpha\gamma y^{-\delta})}{(1+\gamma y^{-\delta})}$. We notice that X and Y are independent if $\alpha = 1$. Indeed, from (2) if $\alpha = 1$ then $f_{Y=y}(x|y; \boldsymbol{\xi}) = f_X(x; \boldsymbol{\xi}_1)$, since $k_y = 1$, where $\boldsymbol{\xi}_1 = (\beta, \lambda, \theta)$.

It is easy to verify that a random variable X given $Y \leq y$ is Burr III distributed, with conditional density function and conditional distribution function, respectively, given by

$$f_{Y\leq y}(x|y; \boldsymbol{\xi}) = \beta k_y\lambda\theta x^{-\theta-1}(1 + k_y\lambda x^{-\theta})^{-\beta-1} \quad (4)$$

$$F_{Y\leq y}(x|y; \boldsymbol{\xi}) = (1 + k_y\lambda x^{-\theta})^{-\beta}. \quad (5)$$

In oder to study the dependence between X and Y, we use the following definition.

Definition 1 *A non-negative function g defined on $\mathbb{R}^2$ is totally positive of order 2, if for all $x_1 < x_2$, $y_1 < y_2$, with x_i, $y_j \in \mathbb{R}$, it holds that $g(x_1, y_1)g(x_2, y_2) \geq g(x_2, y_1)g(x_1, y_2)$ (see Joe, 1997). If the inequality is reversed then g is reverse rule of order 2 ($RR2$).*

We highlight that if g is the joint density function of random vector (X, Y) then $TP2$ coincides with the positively likelihood ratio property of Lehmann (1966). The $TP2$ is a notion of positive dependence and is the strongest of all dependence notions in the literature; for a deep discussion on dependence see, for example, Joe (1997). Holland and Wang (1987) proved the following theorem useful for verify whether a bivariate density function is $TP2$

Theorem 2 *The density of a random vector (X, Y) is $TP2$ if $\gamma_f(x, y) > 0$, where $\gamma_f(x, y) = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y}$ is called local dependence function.*

For the bivariate Rodriguez-Burr III density, it can be verified that

$$\gamma_f(x,y) = (1-\alpha)\theta\gamma\delta x^{-\theta-1}y^{-\delta-1}\left\{\frac{\beta+2}{A^2}+\frac{\alpha^2\beta}{B^2}\right\} \tag{6}$$

where $A = \left(1+\alpha\lambda\gamma x^{-\theta}y^{-\delta}+\lambda x^{-\theta}+\gamma y^{-\delta}\right)$ and $B = \left\{(\beta+1)(1+\alpha\lambda x^{-\theta})(1+\alpha\gamma y^{-\delta})-\alpha A\right\}$. Thus, $f_{XY}(x,y)$ is $TP2$ if $\alpha < 1$. This property will be used in studying the monotonicity of certain failure rates.

3 Some definition and background

In this section, we report the definitions of hazard rate and reversed hazard rate in the bivariate setting and some notions of dependence that we will use throughout the paper. Let (X,Y) be a two dimensional random variable with probability density function $f(x,y)$, distribution function $F(x,y)$ and survival function $S(x,y)$. It is well-known that the hazard gradient (Johnson and Kotz, 1975) is: $\mathbf{h}_{1,2}(x,y) = (h_1(x,y), h_2(x,y))$ where

$$h_1(x,y) = h_{Y>y}(x|y) = -\frac{\partial \ln S(x,y)}{\partial x} \qquad and \qquad h_2(x,y) = h_{X>x}(y|x) = -\frac{\partial \ln S(x,y)}{\partial y}.$$

Moreover, the bivariate reversed hazard rate as a vector (Roy, 2002) is: $\mathbf{rh}_{1,2}(x,y) = (rh_1(x,y), rh_2(x,y))$, where

$$rh_1(x,y) = rh_{Y<y}(x|y) = \frac{\partial \ln F(x,y)}{\partial x} \qquad and \qquad rh_2(x,y) = rh_{X<x}(y|x) = \frac{\partial \ln F(x,y)}{\partial y}.$$

$rh_1(x,y)\Delta x$ is the probability of failure of the first component in the interval $(x-\Delta x, x)$ given that it has failed before x and the second component has failed before y. The intepretation of $rh_2(x,y)$ is similar.

In the final part of this section, in order to study the dependence between X and Y, we recall some notions of dependence. The random vector (X,Y) is said to be left corner set decreasing ($LCSD$) if $P\left(X < x, Y < y | X < x', Y < y'\right)$ is decreasing in x' and y' for all x, y. Analogous to the Shaked (1977) for $RCSI$ (right corner set increasing), Domma (2009 b) proved that the random vector (X,Y) is $LCSD$ if and only if $rh_1(x,y)$ is increasing in y for all x and $rh_2(x,y)$ is increasing in x, for all y. Moreover, X and Y are said to be positively (negatively) quadrant dependent (PQD (NQD)) if $P(X < x, Y < y) > (<) P(X < x)P(Y < y)$; see Lehaman (1966) and Joe (1997). Finally, Y is said to be left tail decreasing in X, $LTD(Y|X)$, if $P(Y < y|X < x)$ is decreasing in x for all y.
For the aims of this work, it is worthwhile pointing out the following relationships among dependence properties. If the joint density function, $f_{XY}(x,y)$ is $TP2$ then (X,Y) is $LCSD$ and $RCSI$; the

bivariate distribution function $F_{XY}(x,y)$ is $TP2$ if and only if (X,Y) is $LCSD$, the bivariate survival function $S_{XY}(x,y)$ is $TP2$ if and only if (X,Y) is $RCSI$. Moreover, it is well known that $LCSD$ implies both $LTD(Y|X)$ and $LTD(X|Y)$, but $LTD(Y|X)$ and $LTD(X|Y)$ taken together do not imply $LCSD$; likewise, $RCSI$ implies both $RTI(Y|X)$ and $RTI(X|Y)$. Finally, it can be easily seen that both $LTD(Y|X)$ and $RTI(Y|X)$ imply PQD; for more details see Joe (1997) and Nelsen (1999).

4 Reliability functions of conditional distributions

In this section, we study some functions used in reliability theory based on the conditional distributions of a random variable X given $Y=y$, of X given $Y<y$ and X given $Y>y$. In particular, using some notions of dependence, we analyse the behaviour of the reversed hazard rate, hazard rate, reversed mean residual lifeme and mean residual life for bivariate Rodriguez-Burr III distribution.

From (2) and (3), the reversed hazard function of X given $Y=y$ is given by

$$rh_{Y=y}(x|y;\boldsymbol{\xi}) = \frac{\lambda\theta x^{-\theta-1}\left[k_y(\beta+1)(1+\alpha\lambda x^{-\theta})-\alpha(1+k_y\lambda x^{-\theta})\right]}{(1+k_y\lambda x^{-\theta})(1+\alpha\lambda x^{-\theta})}.$$

In order to study the monotonicity of $rh_{Y=y}(x|y;\boldsymbol{\xi})$ as function of y, it is easy to verify that

$$\frac{\partial rh_{Y=y}(x|y;\boldsymbol{\xi})}{\partial y} = \frac{(1-\alpha)(\beta+1)\lambda\theta\gamma\delta x^{-\theta-1}y^{-\delta-1}}{(1+k_y\lambda x^{-\theta})^2(1+\gamma y^{-\delta})^2}$$

is greater than zero if and only if $\alpha<1$. Similarly, the reversed hazard rate of the conditional distribution of Y given $X=x$ is increasing if and only if $\alpha<1$.

To calculate the mean reversed residual life of X given $Y=y$, defined as: $\mu r_{Y=y}(x|y;\boldsymbol{\xi}) = \frac{\int_0^x F(u/y;\boldsymbol{\xi})du}{F(x/y;\boldsymbol{\xi})}$, we consider the following

$$I=\int_0^x F_{Y=y}(u|y;\boldsymbol{\xi})du = \int_0^x \left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}du + \alpha\lambda\int_0^x u^{-\theta}\left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}du.$$

By simple manipulation, we obtain

$$\int_0^x u^{-\theta}\left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}du = \frac{(k_y\lambda)^{\frac{1}{\theta}-1}}{\theta}B\left(w^*;\beta+\frac{1}{\theta},1-\frac{1}{\theta}\right)$$

where $B\left(w^*;p,q\right)=\int_0^{w^*}y^{p-1}(1-y)^{q-1}dy$, with $w^*=\left(1+k_y\lambda x^{-\theta}\right)^{-1}<1$.

In order to calculate $\int_0^x\left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}du$, we observe that Domma et al. (2009) proved that if the random variable W is Burr III distributed with distribution function $F_W(w;\epsilon_1,\epsilon_2,\epsilon_3) =$

$\left(1+\epsilon_2 w^{-\epsilon_3}\right)^{-\epsilon_1}$ then the mean reversed residual life is:

$$\mu r\left(w;\epsilon_1,\epsilon_2,\epsilon_3\right)=\frac{\int_0^w F_W(u;\epsilon_1,\epsilon_2,\epsilon_2)du}{F_W(u;\epsilon_1,\epsilon_2,\epsilon_2)}=w-\frac{\epsilon_1\epsilon_2^{\frac{1}{\epsilon_3}}B\left(z^*;\epsilon_1+\frac{1}{\epsilon_3},1-\frac{1}{\epsilon_3}\right)}{F_W(w;\epsilon_1,\epsilon_2,\epsilon_3)}$$

with $z^*=\left(1+\epsilon_2 w^{-\epsilon_3}\right)^{-1}<1$. Now, the function $\left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}$ can be seen as the distribution function of a Burr III random variable with $\epsilon_1=\beta+1$, $\epsilon_2=k_y\lambda$ and $\epsilon_3=\theta$. Therefore, we can write

$$\int_0^x\left(1+k_y\lambda u^{-\theta}\right)^{-(\beta+1)}du=\mu r\left(x;\beta+1,k_y\lambda,\theta\right)\times F\left(x;\beta+1,k_y\lambda,\theta\right).$$

Finally, the mean reversed residual life of X given $Y=y$ is given by:

$$\mu r_{Y=y}(x|y;\boldsymbol{\xi})=\frac{\mu r\left(x;\beta+1,k_y\lambda,\theta\right)F\left(x;\beta+1,k_y\lambda,\theta\right)}{F_{Y=y}(x|y;\boldsymbol{\xi})}+\frac{\alpha\lambda^{\frac{1}{\theta}}k_y^{\frac{1}{\theta}-1}B\left(w^*;\beta+\frac{1}{\theta},1-\frac{1}{\theta}\right)}{\theta F_{Y=y}(x|y;\boldsymbol{\xi})}.$$

The hazard rate and the mean residual life of the random variable X given $Y=y$, denoted with $h_{Y=y}(x|y;\boldsymbol{\xi})$ and $\mu_{Y=y}(x|y;\boldsymbol{\xi})$ respectively, are a complicated function of x. However, their monotonicity as function of y can be determined by employing the following result due to Shaked (1977).

Lemma 3 *If $f_{XY}(x,y)$ is $TP2$ then the hazard rate of X given $Y=y$ is decreasing in y for all x and the mean residual life of X given $Y=y$ is increasing in y for all x.*

Using the above result, we can say that if $\alpha<1$ then $h_{Y=y}(x|y;\boldsymbol{\xi})$ is decreasing in y for all x and $\mu_{Y=y}(x|y;\boldsymbol{\xi})$ is increasing in y for all x.

From (4) and (5), the reversed hazard function of X given $Y<y$ is given by

$$rh_{Y<y}(x|y;\boldsymbol{\xi})=\frac{\partial\ln F(x,y)}{\partial x}=\frac{\beta k_y\lambda\theta x^{-\theta-1}}{(1+k_y\lambda x^{-\theta})}.$$

Using the fact that the random variable $X|Y<y$ is Burr III distributed with parameters β, $k_y\lambda$ and θ, we can say that the mean reversed residual life of X given $Y<y$ is

$$\mu r_{Y<y}\left(x;\beta,k_y\lambda,\theta\right)=x-\frac{\beta(k_y\lambda)^{\frac{1}{\theta}}B\left(z^*;\beta+\frac{1}{\theta},1-\frac{1}{\theta}\right)}{F_{Y<y}(x;\beta,k_y\lambda,\theta)}.$$

Now, provided that the random variable $X|Y<y$ is Burr III distributed, then the reversed hazard function and the mean reversed residual life as function of x is described in Domma et al. (2009).

In order to study the monotonicity of $rh_{Y<y}(x|y;\xi)$ as function of y, we recall that if $f_{XY}(x,y)$ is $TP2$ then (X,Y) is $LCSD$. Using the result by Domma (2009 b), we can conclude that if $\alpha<1$ then $rh_{Y<y}(x|y;\xi)$ is increasing in y for all x.

The hazard rate and the mean residual life of the conditional distribution of X given $Y>y$, denoted by $h_{Y>y}(x|y)$ and $\mu_{Y>y}(x|y)$ respectively, as function of x show a complicate form. However, their monotonicity as a function of y can be determined by employing the following result due to Shaked (1977).

Lemma 4 *If $f_{XY}(x,y)$ is $TP2$ then the hazard rate of X given $Y>y$ is decreasing in y for all x and the mean residual life of X given $Y>y$ is increasing in y for all x.*

Using this result, we can conclude that if $\alpha<1$ then $h_{Y>y}(x|y;\xi)$ is decreasing in y for all x and $\mu_{Y>y}(x|y;\xi)$ is increasing in y for all x.

The distribution of maximum of two random variables X and Y play an important role in various statistical applications. For example, in reliability studies, $T=max(X,Y)$ is observed if the components are arranged in a parallel system. In the final part of this section, using the copula approach, we study the effect of the dependence parameter on the reversed hazard rate of the random variable T. By Sklar's theorem (Sklar, 1959), the Rodriguez-Burr III copula function is:

$$C_{XY}(u,v)=\left\{1+\alpha\left(u^{-\frac{1}{\beta}}-1\right)\left(v^{-\frac{1}{\beta}}-1\right)+\left(u^{-\frac{1}{\beta}}-1\right)+\left(v^{-\frac{1}{\beta}}-1\right)\right\}^{-\beta} \tag{7}$$

where $u,v\in[0,1]\times[0,1]$, see Domma (2009 a). Let $T=max(X,Y)$ be the maximum in a random sample of size two from (1). Then, the distribution function of T is

$$F_T(t,\boldsymbol{\xi})=Pr\left\{max\left(X,Y\right)\leq t\right\}=C_{XY}\left(F_X(t,\boldsymbol{\xi_1}),F_Y(t,\boldsymbol{\xi_2});\alpha\right).$$

Moreover, if $F_X(x;\boldsymbol{\xi_1})$ and $F_Y(y;\boldsymbol{\xi_2})$ are identical then

$$F_T(t,\boldsymbol{\xi}_1;\alpha)=\delta_C\left(F_X(t;\boldsymbol{\xi}_1);\alpha\right) \tag{8}$$

where $\delta_C\left(F_X(t;\boldsymbol{\xi}_1;\alpha)\right)=C\left(F_X(t;\boldsymbol{\xi}_1),F_X(t;\boldsymbol{\xi}_1);\alpha\right)$ is the diagonal section of copula $C(.;.)$, see Nelsen (1999). Therefore, it is simple to verify that the diagonal section of copula function (7) is

$$\begin{aligned} C_{XY}\left(F_X(t;\boldsymbol{\xi}_1),F_X(t;\boldsymbol{\xi}_1);\alpha\right) &= \left\{1+\alpha\left([F_X(t;\boldsymbol{\xi}_1)]^{-\frac{1}{\beta}}-1\right)^2+2\left([F_X(t;\boldsymbol{\xi}_1)]^{-\frac{1}{\beta}}-1\right)\right\}^{-\beta} \\ &= \left\{1+\alpha\lambda^2t^{-2\theta}+2\lambda t^{-\theta}\right\}^{-\beta}. \end{aligned}$$

Therefore, the distribution function and density function of T, respectively, are

$$F_T(t;\boldsymbol{\xi}_1;\alpha)=\left\{1+\alpha\lambda^2t^{-2\theta}+2\lambda t^{-\theta}\right\}^{-\beta} \tag{9}$$

and

$$f_T(t;\boldsymbol{\xi}_1;\alpha) = 2\beta\lambda\theta t^{-\theta-1}\left(1+\alpha\lambda t^{-\theta}\right)\left\{1+\alpha\lambda^2 t^{-2\theta}+2\lambda t^{-\theta}\right\}^{-\beta-1} \tag{10}$$

Moreover, the reversed hazard rate of T is

$$rh_T(t;\boldsymbol{\xi}_1;\alpha) = \frac{f_T(t;\boldsymbol{\xi}_1;\alpha)}{F_T(t;\boldsymbol{\xi}_1;\alpha)} = rh_T^{\perp}(t;\boldsymbol{\xi}_1) \times \frac{\left[1+(1+\alpha)\lambda t^{-\theta}+\alpha\lambda^2 t^{-2\theta}\right]}{\left[1+2\lambda t^{-\theta}+\alpha\lambda^2 t^{-2\theta}\right]} \tag{11}$$

where $rh_T^{\perp}(t;\boldsymbol{\xi}_1) = \frac{2\beta\lambda\theta t^{-\theta-1}}{\left(1+\lambda t^{-\theta}\right)}$ is the reversed hazard rate of T when X and Y are independent. By (11) it is simple to verify that if $\alpha \leq 1$ then $rh_T(t;\boldsymbol{\xi}_1;\alpha) \leq rh_T^{\perp}(t;\boldsymbol{\xi}_1)$ and if $\alpha > 1$ then $rh_T(t;\boldsymbol{\xi}_1;\alpha) > rh_T^{\perp}(t;\boldsymbol{\xi}_1)$.

5 Dependence measures

Analogous to the Clayton (1978) and Oakes (1989) association measures based on cross ratios of bivariate survival functions, Sankaran and Gleeja (2006) defined a local dependence measure in terms of a bivariate distribution function, given by

$$\lambda(x,y) = \frac{F_{12}F}{F_1F_2}$$

where $F = F(x,y)$, $F_{12} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$, $F_1 = \frac{\partial F(x,y)}{\partial x}$ and $F_2 = \frac{\partial F(x,y)}{\partial y}$. The symbol $\lambda(x,y)$ can be interpreted as the ratio of the reversed hazard ratio of the conditional distribution of X given $Y = y$ to that of X given $Y < y$, i.e. $\lambda(x,y) = \frac{rh_{Y=y}(x|y)}{rh_{Y<y}(x|y)}$; by simmetry a similar interpretation holds with (X,Y) interchanged. Sankaran and Gleeja (2006) proved that $\lambda(x,y) = 1$ if and only if X and Y are independent. Moreover, Sankaran and Gleeja (2008) declare that (X,Y) is $LCSD$ if $\lambda(x,y) > 1$. Domma (2009 b) provided a stronger result about this measure having proved that $\lambda(x,y) > 1$ if and only if (X,Y) is $LCSD$. In particular, he proved the following

Proposition 5 *The following statments are equivalent:*

i) $\lambda(x,y) > 1$

ii) $\frac{\partial^2 \ln F(x,y)}{\partial x \partial y} > 0$

iii) *(X,Y) is $LCSD$.*

Recalling that the bivariate Rodriguez-Burr III density is $TP2$ if $\alpha < 1$ and that $TP2$ implies $LCSD$ then we can conclude that for this model $\lambda(x,y) > 1$ if and only if $\alpha < 1$. On the other hand, it is easy to verify that if $\alpha > 1$ then $\lambda(x,y) < 1$. In fact, we have

$$\lambda(x,y) = \frac{1}{\beta}\left\{(\beta+1) - B\right\}$$

where $B = \frac{\alpha\left(1+\alpha\lambda\gamma x^{-\theta}y^{-\delta}+\lambda x^{-\theta}+\gamma y^{-\delta}\right)}{\left(1+\alpha\lambda x^{-\theta}\right)\left(1+\alpha\gamma y^{-\delta}\right)}$.
Thus, $\lambda(x, y) > (<)1$ if and only if $(1 - B) > (<)0$ and this holds if and only if $\alpha < 1(> 1)$. $\square$

Analogous to the $\phi_1(x, y)$ and $\Psi(x, y)$ measures proposed by Anderson et al. (1992), Sankaran and Gleeja (2008) defined two dependence measures based on the mean reversed residual life and bivariate distribution function, respectively, given by

$$\bar{\phi}_1(x, y) = \frac{E\left(x - X|X < x, Y < y\right)}{E\left(x - X|X < x\right)} = \frac{\mu r_{Y<y}(x, y)}{\mu r(x)}$$

and

$$\bar{\Psi}(x, y) = \frac{P\left(X < x|Y < y\right)}{P\left(X < x\right)} = \frac{F(x, y)}{F(x, +\infty)F(+\infty, y)}.$$

Moreover, they proved that $\bar{\phi}_1(x, y) = \bar{\Psi}(x, y) = 1$ if and only if X and Y are independent and if $\lambda(x, y) > 1$ then $\bar{\phi}_1(x, y) > 1$.

In order to compute the dependence measure $\bar{\phi}_1(x, y)$ for the bivariate Rodriguez-Burr III distribution, we preliminarily calculate the following expression:

$$\begin{aligned} \int_0^x F(u, y)du &= \left(1 + \gamma y^{-\delta}\right)^{-\beta} \int_0^x \left(1 + k_y\lambda u^{-\theta}\right)^{-\beta} du = \\ &= F_Y(y; \beta, \gamma, \delta) \int_0^x F_X(u; \beta, k_y\lambda, \theta)du = \\ &= F_Y(y; \beta, \gamma, \delta)F_X(u; \beta, k_y\lambda, \theta)\mu r(x; \beta, k_y\lambda, \theta). \end{aligned}$$

Using this result, we obtain

$$\mu r_1(x, y) = \frac{\int_0^x F(u, y)du}{F(x, y)} = \frac{F_Y(y; \beta, \gamma, \delta)F_X(u; \beta, k_y\lambda, \theta)}{F(x, y)}\mu r(x; \beta, k_y\lambda, \theta) = \mu r(x; \beta, k_y\lambda, \theta)$$

because it is simple to prove that $F_Y(y; \beta, \gamma, \delta)F_X(u; \beta, k_y\lambda, \theta) = F(x, y)$. After all, the measure $\bar{\phi}_1(x, y)$ for the bivariate Rodriguez-Burr III distribution is given by

$$\bar{\phi}_1(x, y) = \frac{\mu r(x; \beta, k_y\lambda, \theta)}{\mu r(x; \beta, \lambda, \theta)}.$$

Finally, we point out that if $\alpha = 1$, $\bar{\phi}_1(x, y) = 1$ because $k_y = 1$.
For the bivariate Rodriguez-Burr III distribution, Domma (2009 a) proved that (X, Y) is PQD (NQD) if and only if $\alpha < (>)1$, hence $\bar{\Psi}(x, y) > (<)1$ if and only if $\alpha < (>)1$; In fact, since the marginal distribution of this model are Burr III, i.e. $F(x, +\infty) = \left(1 + \lambda x^{-\theta}\right)^{-\beta}$ and $F(+\infty, y) = \left(1 + \gamma y^{-\delta}\right)^{-\beta}$, the dependence measures $\bar{\Psi}(x, y)$ is given by

$$\bar{\Psi}(x, y) = \left\{\frac{\left(1 + \lambda x^{-\theta}\right)\left(1 + \gamma y^{-\delta}\right)}{(1 + \alpha\lambda\gamma x^{-\theta}y^{-\delta} + \lambda x^{-\theta} + \gamma y^{-\delta})}\right\}^{\beta},$$

for which it is immediate to deduce that $\bar{\Psi}(x, y) > 1$ if and only if $\alpha < 1$.

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