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THE FISHER INFORMATION MATRIX IN RIGHT CENSORED DATA FROM THE DAGUM DISTRIBUTION

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Abstract

In this note, we provide the mathematical details of the calculation of the Fisher information matrix when the data involve type I right censored observations from a Dagum distribution.

1 The Dagum distribution

The Dagum model (Dagum 1977, 1980) has been successfully used in studies on income and wealth distributions, the reader refers to Kleiber and Kotz (2003) and Kleiber (2007) for an excellent survey on the genesis and practical applications of the Dagum model.

The random variable T, continuous and non negative, is Dagum distributed if its cumulative distribution function (*cdf*) is

$$F_T(t;\boldsymbol{\theta}) = \left(1+\lambda t^{-\delta}\right)^{-\beta} \tag{1}$$

and the probability density function (*pdf*) is

$$f_T(t;\boldsymbol{\theta}) = \beta\lambda\delta t^{-\delta-1}\left(1+\lambda t^{-\delta}\right)^{-\beta-1} \tag{2}$$

where $\boldsymbol{\theta} = (\beta, \lambda, \delta)$, and $\beta > 0$, $\lambda > 0$ and $\delta > 0$. The parameter λ is a scale parameter, while β and δ are shape parameters. It is well-known that the Dagum distribution has positive asymmetry, it is unimodal for $\beta\delta > 1$ and zero-modal for $\beta\delta \leq 1$. Moreover, it is easy to verify that the *q-th* quantile of the Dagum distribution is $t(q) = \lambda^{\frac{1}{\delta}} \cdot (q^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\delta}}$, whereas the *r-th* moment is:

$$E(T^r|\beta,\lambda,\delta) = \beta\lambda^{\frac{r}{\delta}} B\left(\beta+\frac{r}{\delta}, 1-\frac{r}{\delta}\right) \tag{3}$$

for $\delta > r$, where $B(.,.)$ is the mathematical function Beta.

2 Maximum Likelihood Estimation

In this section, the maximum likelihood estimation (MLE) of the parameters β, λ and δ of the Dagum distribution under type I censoring mechanism is presented.

Consider a sample of size n of independent positive random variables $T_1, ..., T_n$ such that T_i has associated an indicator variable ϵ_i, where $\epsilon_i = 1$ if T_i is an observed failure time and $\epsilon_i = 0$ if T_i is right censored. Let $\boldsymbol{\theta} = (\beta, \lambda, \delta)$, the log-likelihood function based on data (t_1, ϵ_1), .., (t_n, ϵ_n) is

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \epsilon_i \left\{ \ln(\beta\lambda\delta) - (\delta+1)\ln(t_i) - (\beta+1)\ln\left(1+\lambda t_i^{-\delta}\right) \right\} + \sum_{i=1}^{n} (1-\epsilon_i) \ln \left\{ 1 - \left[1+\lambda t_i^{-\delta}\right]^{-\beta} \right\}. \tag{4}$$

The MLEs $\hat{\boldsymbol{\theta}}_n = (\hat{\beta}_n, \hat{\lambda}_n, \hat{\delta}_n)$ are obtained from the maximization of (4), as the solution of the following system of equations:

$$\begin{cases} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} = \sum\limits_{i=1}^{n} \epsilon_i \left\{ \frac{1}{\beta} - \ln\left(1+\lambda t_i^{-\delta}\right) \right\} - \frac{1}{\beta} \sum\limits_{i=1}^{n} \frac{(1-\epsilon_i) F_T(t_i;\boldsymbol{\theta}) \ln[F_T(t_i;\boldsymbol{\theta})]}{S_T(t_i;\boldsymbol{\theta})} = 0 \\ \frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda} = \sum\limits_{i=1}^{n} \epsilon_i \left\{ \frac{1}{\lambda} - (\beta+1) t_i^{-\delta} \left[F_T(t_i;\boldsymbol{\theta})\right]^{\frac{1}{\beta}} \right\} + \beta \sum\limits_{i=1}^{n} \frac{(1-\epsilon_i) t_i^{-\delta} [F_T(t_i;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{S_T(t_i;\boldsymbol{\theta})} = 0 \\ \frac{\partial \ell(\boldsymbol{\theta})}{\partial \delta} = \sum\limits_{i=1}^{n} \epsilon_i \left\{ \frac{1}{\delta} - \ln(t_i) + \lambda(\beta+1)\ln(t_i) t_i^{-\delta} \left[F_T(t_i;\boldsymbol{\theta})\right]^{\frac{1}{\beta}} \right\} - \beta\lambda \sum\limits_{i=1}^{n} \frac{(1-\epsilon_i)\ln(t_i) t_i^{-\delta} [F_T(t_i;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{S_T(t_i;\boldsymbol{\theta})} = 0. \end{cases}$$

The system does not admit any explicit solution, therefore the ML estimates $\hat{\boldsymbol{\theta}}_n = (\hat{\beta}_n, \hat{\lambda}_n, \hat{\delta}_n)$ can be obtained only by means of numerical procedures.

Under the usual regularity conditions, the well-known asymptotic properties of the maximum likelihood method ensure that $\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right)$, where $\boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \left[\mathbf{I}(\boldsymbol{\theta})\right]^{-1}$ is the asymptotic variance-covariance matrix and $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher Information Matrix, whose entries will be calculated in the next section.

3 Fisher Information Matrix

In order to derive the elements of the Fisher Information Matrix, we preliminary obtain the following second partial derivatives of the log-likelihood function:

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta^2} = -\frac{1}{\beta^2} \sum_{i=1}^{n} \left\{ \epsilon_i + (1-\epsilon_i) \left[\frac{\ln F_T(t_i;\boldsymbol{\theta})}{S_T(t_i;\boldsymbol{\theta})} \right]^2 F_T(t_i;\boldsymbol{\theta}) \right\}$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \partial \lambda} = \sum_{i=1}^{n} \left\{ S_T(t_i;\boldsymbol{\theta}) \left(\frac{\epsilon_i}{F_T(t_i;\boldsymbol{\theta})} + 1 \right) + (1-\epsilon_i) \ln F_T(t_i;\boldsymbol{\theta}) \right\} \frac{F_T(t_i;\boldsymbol{\theta})^{1+\frac{1}{\beta}} T^{-\delta}}{S_T(t_i;\boldsymbol{\theta})^2}$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \beta \partial \delta} = -\lambda \sum_{i=1}^{n} \left\{ S_T(t_i;\boldsymbol{\theta}) \left(\frac{\epsilon_i}{F_T(t_i;\boldsymbol{\theta})} + 1 \right) + (1-\epsilon_i) \ln F_T(t_i;\boldsymbol{\theta}) \right\} \frac{F_T(t_i;\boldsymbol{\theta})^{1+\frac{1}{\beta}} T^{-\delta} \ln(T)}{S_T(t_i;\boldsymbol{\theta})^2}$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda^2} = -\frac{1}{\lambda^2} \sum_{i=1}^{n} \epsilon_i - \sum_{i=1}^{n} \left\{ \beta(1-\epsilon_i) \frac{F_T(t_i;\boldsymbol{\theta})}{S_T(t_i;\boldsymbol{\theta})} \left(1 - \frac{\beta}{S_T(t_i;\boldsymbol{\theta})} \right) - (\beta+1)\epsilon_i \right\} F_T(t_i;\boldsymbol{\theta})^{\frac{2}{\beta}} T^{-2\delta}$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \delta} = -\sum_{i=1}^{n} \left\{ \beta(1-\epsilon_i)\frac{F_T(t_i;\boldsymbol{\theta})}{S_T(t_i;\boldsymbol{\theta})} \left(1 - \frac{\beta}{S_T(t_i;\boldsymbol{\theta})}\right) - (\beta+1)\epsilon_i \right\} F_T(t_i;\boldsymbol{\theta})^{\frac{2}{\beta}} T^{-2\delta} \ln(T)$$

$$\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta^2} = -\frac{1}{\delta^2}\sum_{i=1}^{n}\epsilon_i + \sum_{i=1}^{n} \left\{ \beta(1-\epsilon_i)\frac{F_T(t_i;\boldsymbol{\theta})}{S_T(t_i;\boldsymbol{\theta})} \left(1 - \frac{\beta\lambda T^{-\delta}}{S_T(t_i;\boldsymbol{\theta})}\right) - (\beta+1)\epsilon_i \right\} F_T(t_i;\boldsymbol{\theta})^{\frac{2}{\beta}} T^{-\delta} \left[\ln(T)\right]^2.$$

To compute the entries of the Fisher information matrix we need the following expectations

$$\begin{aligned}
E^{(1)} &= E\left\{ \frac{[F(T;\boldsymbol{\theta})]}{[S(T;\boldsymbol{\theta})]^2} \left[\ln F(T;\boldsymbol{\theta})\right]^2 \right\}, \quad E^{(2)} = E\left\{ \frac{[F(T;\boldsymbol{\theta})]^{\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-\delta} \right\} \\
E^{(3)} &= E\left\{ \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-\delta} \right\}, \quad E^{(4)} = E\left\{ \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-2\delta} \right\} \\
E^{(5)} &= E\left\{ \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]^2} T^{-\delta} \ln\left[F(T;\boldsymbol{\theta})\right] \right\}, \quad E^{(6)} = E\left\{ \ln(T) \frac{[F(T;\boldsymbol{\theta})]^{\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-\delta} \right\} \\
E^{(7)} &= E\left\{ \ln(T) \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-\delta} \right\}, \quad E^{(8)} = E\left\{ \ln(T) \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{1}{\beta}}}{[S(T;\boldsymbol{\theta})]^2} T^{-\delta} \ln\left[F(T;\boldsymbol{\theta})\right] \right\} \\
E^{(9)} &= E\left\{ \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]^2} T^{-2\delta} \right\}, \quad E^{(10)} = E\left\{ [F(T;\boldsymbol{\theta})]^{\frac{2}{\beta}} T^{-2\delta} \right\} \\
E^{(11)} &= E\left\{ \ln(T) [F(T;\boldsymbol{\theta})]^{\frac{2}{\beta}} T^{-2\delta} \right\}, \quad E^{(12)} = E\left\{ \ln(T) \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]^2} T^{-2\delta} \right\} \\
E^{(13)} &= E\left\{ \ln(T) \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-2\delta} \right\}, \quad E^{(14)} = E\left\{ [\ln(T)]^2 \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]} T^{-\delta} \right\} \\
E^{(15)} &= E\left\{ [\ln(T)]^2 \frac{[F(T;\boldsymbol{\theta})]^{1+\frac{2}{\beta}}}{[S(T;\boldsymbol{\theta})]^2} T^{-2\delta} \right\}, \quad E^{(16)} = E\left\{ [\ln(T)]^2 [F(T;\boldsymbol{\theta})]^{\frac{2}{\beta}} T^{-\delta} \right\}.
\end{aligned}$$

We highlight that $E^{(1)}, ..., E^{(16)}$ are special cases of the following expectation

$$E_{k_1,k_2,k_3,k_4,k_5,k_6} = E\left\{ [\ln(T)]^{k_1} \frac{[F_T(t_i;\boldsymbol{\theta})]^{k_2+\frac{k_3}{\beta}}}{[S_T(t_i;\boldsymbol{\theta})]^{k_4}} T^{-\delta k_5} \left[\ln\left(F_T(t_i;\boldsymbol{\theta})\right)\right]^{k_6} \right\}$$

with k_1, k_2, k_3, k_4, k_5 and k_6 positive integers. Now, setting $y = \left(1 + \lambda t^{-\delta}\right)^{-1}$, after some algebra, we obtain

$$\begin{aligned}
E_{k_1,k_2,k_3,k_4,k_5,k_6} &= \beta^{k_6+1}\lambda^{-k_5} \int_0^1 \left[\ln\left\{ \lambda^{\frac{1}{\delta}} \left(\frac{y}{1-y}\right)^{\frac{1}{\delta}} \right\} \right]^{k_1} \times \\
&\quad [\ln(y)]^{k_6} \left(1 - y^{\beta}\right)^{k_4} y^{\beta(k_6+1)+k_3-k_5-1} (1-y)^{k_5}\, dy.
\end{aligned}$$

Setting $k_1 = 0$, $k_2 = 1$, $k_3 = 0$, $k_4 = 2$, $k_5 = 0$, $k_6 = 2$ and $k_1 = 0$, $k_2 = 0$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$, $k_6 = 0$ in $E_{k_1,k_2,k_3,k_4,k_5,k_6}$, respectively, we obtain

$$E^{(1)} = E_{0,1,0,2,0,2,0} = \frac{2}{2^3} - \frac{4}{3^3} + -\frac{2}{4^3} \qquad and \qquad E^{(2)} = E_{0,0,1,1,1,0,0} = \frac{\beta\left[B(\beta,2) - B(2,2)\right]}{\lambda}$$

Setting $k_1 = 0$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$, $k_6 = 0$ and $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 1$, $k_5 = 2$, $k_6 = 0$ in $E_{k_1,k_2,k_3,k_4,k_5,k_6}$, respectively, we have

$$E^{(3)} = E_{0,1,1,1,1,0,0} = \frac{\beta\left[B(2\beta,2) - B(3\beta,2)\right]}{\lambda} \quad and \quad E^{(4)} = E_{0,1,2,1,2,0,0} = \frac{\beta\left[B(\beta,3) - B(3\beta,3)\right]}{\lambda^2}$$

For $k_1 = 0$, $k_2 = 1$, $k_3 = 1$, $k_4 = 2$, $k_5 = 1$ and $k_6 = 1$,

$$\begin{aligned} E^{(5)} = E_{0,1,1,2,1,1} \quad = \quad & \frac{\beta^2}{\lambda}\left\{B(2\beta,2)\left[\Psi(2\beta) - \Psi(2\beta+2)\right] - 2B(3\beta,2)\left[\Psi(3\beta) - \Psi(3\beta+2)\right] + \right. \\ & \left. B(4\beta,2)\left[\Psi(4\beta) - \Psi(4\beta+2)\right]\right\} \end{aligned}$$

For $k_1 = 1$, $k_2 = 0$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$ and $k_6 = 0$,

$$E^{(6)} = E_{1,0,1,1,1,0} = \frac{\beta}{\lambda\delta}\left\{\ln(\lambda)\left[B(\beta,2) - B(2\beta,2)\right] + B(\beta,2)\left[\Psi(\beta) - \Psi(2)\right] + B(2\beta,2)\left[\Psi(2) - \Psi(2\beta)\right]\right\}$$

For $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$ and $k_6 = 0$,

$$E^{(7)} = E_{1,1,1,1,1,0} = \frac{\beta}{\lambda\delta}\left\{\ln(\lambda)\left[B(2\beta,2) - B(3\beta,2)\right] + B(2\beta,2)\left[2\Psi(\beta) - \Psi(2)\right] + B(3\beta,2)\left[\Psi(2) - \Psi(3\beta)\right]\right\}$$

For $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 2$, $k_5 = 1$ and $k_6 = 1$,

$$\begin{aligned} E^{(8)} = E_{1,1,1,2,1,1} \quad = \quad & \frac{\beta^2}{\lambda\delta}\left\{\ln(\lambda)\left[I_1(2\beta) - 2I_1(3\beta) + I_1(4\beta)\right] + \left[I_{31}(2\beta) - 2I_3(3\beta) + I_3(4\beta)\right] - \right. \\ & \left. \left[I_3(2\beta) - 2I_5(3\beta) + I_5(4\beta)\right]\right\} \end{aligned}$$

For $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 2$, $k_5 = 2$, $k_6 = 0$ and for $k_1 = 0$, $k_2 = 0$, $k_3 = 2$, $k_4 = 0$, $k_5 = 2$, $k_6 = 0$, we obtain

$$E^{(9)} = E_{0,1,2,2,2,0} = \frac{\beta\left[B(2\beta,3) - 2B(3\beta,3) + B(4\beta,3)\right]}{\lambda^2} \quad and \quad E^{(10)} = E_{0,0,2,0,2,0} = \frac{\beta B(\beta,3)}{\lambda^2}$$

For $k_1 = 1$, $k_2 = 0$, $k_3 = 2$, $k_4 = 0$, $k_5 = 2$, $k_6 = 0$

$$E^{(11)} = E_{1,0,2,0,2,0} = \frac{\beta B(\beta+1,2)\left[\ln(\lambda) + \Psi(\beta+1) - \Psi(2)\right]}{\lambda\delta}$$

For $k_1 = 1$, $k_2 = 1$, $k_3 = 2$, $k_4 = 2$, $k_5 = 2$, $k_6 = 0$

$$\begin{aligned} E^{(12)} = E_{1,1,2,2,2,0} \;&=\; \frac{\beta}{\delta\lambda^2}\{\ln(\lambda)\left[B(2\beta,3) - 2B(3\beta,3) + B(4\beta,3)\right] + \\ &\quad I_6(2\beta) - 2I_6(3\beta) + I_6(4\beta) - I_7(2\beta) + 2I_7(3\beta) - I_7(4\beta)\} \end{aligned}$$

For $k_1 = 1$, $k_2 = 1$, $k_3 = 2$, $k_4 = 1$, $k_5 = 2$, $k_6 = 0$

$$\begin{aligned} E^{(13)} = E_{1,1,2,1,2,0} \;&=\; \frac{\beta}{\delta\lambda}\{\left[\ln(\lambda) - \Psi(2)\right]\left[B(2\beta+1,2) - B(3\beta+1,2)\right] + \\ &\quad B(2\beta+1,2)\Psi(2\beta+1) - B(3\beta+1,2)\Psi(3\beta+1)\} \end{aligned}$$

For $k_1 = 2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 1$, $k_5 = 1$, $k_6 = 0$

$$\begin{aligned} E^{(14)} = E_{2,1,2,1,1,0} \;&=\; \frac{\beta}{\lambda\delta^2}\left\{[\ln(\lambda)]^2\left[B(2\beta+1,2) - B(3\beta+1,2)\right] + \right. \\ &\quad 2\ln(\lambda)\left[I_1(2\beta+1) - I_2(2\beta+1) - I_1(3\beta+1) + I_2(3\beta+1)\right] + \\ &\quad \left.\left[I_3(2\beta+1) + I_4(2\beta+1) - 2I_5(2\beta+1) - I_3(3\beta+1) - I_4(3\beta+1) + 2I_5(3\beta+1)\right]\right\} \end{aligned}$$

For $k_1 = 2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 2$, $k_5 = 2$, $k_6 = 0$

$$\begin{aligned} E^{(15)} = E_{2,1,2,2,2,0} \;&=\; \frac{\beta}{\lambda^2\delta^2}\{\ln(\lambda)\left[B(2\beta,3) - 2B(3\beta,3) + B(4\beta,3)\right] + \\ &\quad 2ln(\lambda)\left[I_6(2\beta) - 2I_6(3\beta) + I_6(4\beta) - I_7(2\beta) + 2I_7(3\beta) - I_7(4\beta)\right] \\ &\quad I_9(2\beta) - 2I_9(3\beta) + I_9(4\beta) + I_{10}(2\beta) - 2I_{10}(3\beta) + I_{10}(4\beta) \\ &\quad -2\left[I_8(2\beta) - 2I_8(3\beta) + I_8(4\beta)\right]\} \end{aligned}$$

For $k_1 = 2$, $k_2 = 0$, $k_3 = 2$, $k_4 = 0$, $k_5 = 1$, $k_6 = 0$

$$\begin{aligned} E^{(16)} = E_{2,0,2,0,1,0} \;&=\; \frac{\beta}{\lambda\delta}\{\ln(\lambda)B(\beta+1,2)\left[\ln(\lambda) + 2\Psi(\beta+1) - 2\Psi(2)\right] \\ &\quad + \left[I_3(\beta+1) + I_4(\beta+1) - 2I_5(\beta+1)\right]\} \end{aligned}$$

where the integrals $I_1(.)$, $I_2(.)$, ..., $I_{10}(.)$ are reported in Section 4.

Now, using the expectations $E^{(1)}$, ..., $E^{(16)}$, it is simple to calculate the entries of the Fisher information matrix

$$I_{\beta\beta}(\boldsymbol{\theta}) = \frac{1}{\beta^2}\sum_{i=1}^{n}\left\{\epsilon_i + (1-\epsilon_i)E^{(1)}\right\}$$

$$I_{\beta\lambda}(\boldsymbol{\theta}) = -\sum_{i=1}^{n}\left\{\epsilon_i E^{(2)} + E^{(3)} + (1-\epsilon_i)E^{(5)}\right\}$$

$$I_{\beta\delta}(\boldsymbol{\theta}) = \lambda\sum_{i=1}^{n}\left\{\epsilon_i E^{(6)} + E^{(7)} + (1-\epsilon_i)E^{(8)}\right\}$$

$$I_{\lambda\lambda}(\boldsymbol{\theta}) = \tfrac{1}{\lambda^2}\sum_{i=1}^{n}\{\epsilon_i + \beta(1-\epsilon_i)E^{(4)} - \beta^2(1-\epsilon_i)E^{(9)} - (\beta+1)\epsilon_i E^{(10)}\}$$

$$I_{\lambda\delta}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\{\beta(1-\epsilon_i)E^{(13)} - \beta^2(1-\epsilon_i)E^{(12)} - (\beta+1)\epsilon_i E^{(11)}\}$$

$$I_{\delta\delta}(\boldsymbol{\theta}) = \tfrac{1}{\delta}\sum_{i=1}^{n}\epsilon_i - \sum_{i=1}^{n}\{\beta(1-\epsilon_i)E^{(14)} - \lambda\beta^2(1-\epsilon_i)E^{(15)} - (\beta+1)\epsilon_i E^{(16)}\}$$

4 Some useful notation

Here, for $p > 0$, we report the following expressions above mentioned

$$I_1(p+1) = \int_0^1 z^p(1-z)\ln(z)dz = B\left(p+1,2\right)\left[\Psi(p+1)-\Psi(p+3)\right]$$

$$I_2(p+1) = \int_0^1 z^p(1-z)\ln(1-z)dz = B\left(2,p+1\right)\left[\Psi(2)-\Psi(p+3)\right]$$

$$\begin{aligned} I_3(p+1) &= \int_0^1 z^p(1-z)\left[\ln(z)\right]^2dz = B\left(p+1,2\right)\left\{\left[\Psi(p+1)-\Psi(p+3)\right]^2 + \right. \\ &\quad \left. +\Psi'(p+1)-\Psi'(p+3)\right\} \end{aligned}$$

$$
\begin{aligned}
I_4(p+1) &= \int_0^1 z^p(1-z)\left[\ln(1-z)\right]^2 dz = B\left(2,p+1\right)\left\{\left[\Psi(2)-\Psi(p+3)\right]^2+\right. \\
&\left. +\Psi'(2)-\Psi'(p+3)\right\}
\end{aligned}
$$

$$
I_5(p+1) = \int_0^1 z^p(1-z)\ln(z)\ln(1-z)dz = I(p+1)-I(p+2)
$$

where

$$
I(q+1) = \int_0^1 z^q \ln(z)\ln(1-z)dz = B(q+1,1)\left\{B(q+1,1)\left[\Psi(q+2)-\Psi(2)\right]-\Psi'(q+2)\right\}
$$

for any $q>0$, where $\Psi\left(.\right)$ and $\Psi'\left(.\right)$ are the digamma and trigamma functions, respectively. Moreover, using the integration by part, we have

$$
I_6(p+1) = \int_0^1 z^p(1-z)^2\ln(z)dz = B(p+1,1)\left\{2I_1(p+1)-B(p+1,3)\right\}
$$

$$
I_7(p+1) = \int_0^1 z^p(1-z)^2\ln(1-z)dz = B(p+1,1)\left\{2I_2(p+2)+B(p+2,2)\right\}
$$

$$
\begin{aligned}
I_8(p+1) &= \int_0^1 z^p(1-z)^2\ln(z)\ln(1-z)dz = B(p+1,1)\left\{2I_5(p+2)-\right. \\
&\left. B(p+1,1)\left[2I_2(p+2)-B(p+2,2)\right]+I_1(p+2)\right\}
\end{aligned}
$$

$$
I_9(p+1) = \int_0^1 z^p(1-z)^2\left[\ln(z)\right]^2 dz = \frac{2}{(p+1)^3}-\frac{4}{(p+2)^3}+\frac{2}{(p+3)^3}
$$

$$
I_{10}(p+1) = \int_0^1 z^p(1-z)^2\left[\ln(1-z)\right]^2 dz = 2B(p+1,1)\left\{I_4(p+2)+I_2(p+2)\right\}.
$$

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