Modello di regressione lineare multipla

Y variabile quantitativa (variabile di risposta)

$X_1, X_2, \cdots, X_k$ variabili quantitative (variabili esplicative o regressori)

Modello di relazione (stocastico)

$$Y = f\left(X_1, X_2, \cdots, X_k\right) + e$$

e = variabile casuale *errore* (non osservabile)

errore casuale, disturbo, rumore

Sommatoria degli effetti su Y di una infinità di variabili, non incluse in f

$$f\left(X_1, X_2, \cdots, X_k\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k$$

funzione *lineare* nei parametri $\beta_0, \beta_1, \beta_2, \cdots, \beta_k$

$$\boxed{Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k + e}$$

Modello di regressione lineare

Y variabile casuale osservabile $\qquad X_1, X_2, \cdots, X_k$ variabili deterministiche

Ipotesi (1): $E(e) = 0 \qquad \Rightarrow E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k$

$$\frac{\partial E(Y)}{\partial X_i} = \beta_i \qquad i = 1, \cdots, k$$

Matrice dei dati :

$$\begin{array}{ccccc}
\hline
y_1 & x_{11} & x_{21} & \cdots & x_{k1} \\
y_2 & x_{12} & x_{22} & \cdots & x_{k2} \\
\vdots & \vdots & \vdots & & \vdots \\
y_n & x_{1n} & x_{2n} & \cdots & x_{kn} \\
\hline
\end{array}$$

Modelli statistici:

$$\begin{aligned}
Y_1 &= \beta_0 + \beta_1 x_{11} + \beta_2 x_{21} \cdots + \beta_k x_{k1} + e_1 \\
Y_2 &= \beta_0 + \beta_1 x_{12} + \beta_2 x_{22} \cdots + \beta_k x_{k2} + e_2 \\
&\vdots \\
Y_n &= \beta_0 + \beta_1 x_{1n} + \beta_2 x_{2n} \cdots + \beta_k x_{kn} + e_n
\end{aligned}$$

$$Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_k x_{ki} + e_i \qquad i = 1, \cdots, n$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix}$$

$$p = k + 1$$

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$$

$$\begin{aligned} \mathbf{e}^T \mathbf{e} &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \mathbf{Y}^T\mathbf{Y} - \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y} - \mathbf{Y}^T\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \\ &= \mathbf{Y}^T\mathbf{Y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \\ &= S(\boldsymbol{\beta}) \end{aligned}$$

$\hat{\boldsymbol{\beta}}$ stimatore di *minimi quadrati* di $\boldsymbol{\beta}$ se $S(\hat{\boldsymbol{\beta}}) \le S(\boldsymbol{\beta}) \;\; \forall \boldsymbol{\beta}$

$$S(\boldsymbol{\beta}) = \mathbf{Y}^T\mathbf{Y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

$$\frac{\partial S(\boldsymbol{\beta})}{\boldsymbol{\beta}} = -2\mathbf{X}^T\mathbf{Y} + 2\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \qquad \Rightarrow \; -2\mathbf{X}^T\mathbf{Y} + 2\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{0}$$

$$\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{Y}$$ p equazioni *normali*

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} & \cdots & \sum x_{ki} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i}x_{2i} & \cdots & \sum x_{1i}x_{ki} \\ \vdots & \vdots & \vdots & & \vdots \\ \sum x_{ki} & \sum x_{1i}x_{ki} & \sum x_{2i}x_{kn} & \cdots & \sum x_{ki}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_{1i}Y_i \\ \vdots \\ \sum x_{ki}Y_i \end{bmatrix}$$

$$r(\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = r \le p \qquad \text{se } r = p \quad \hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

unica soluzione

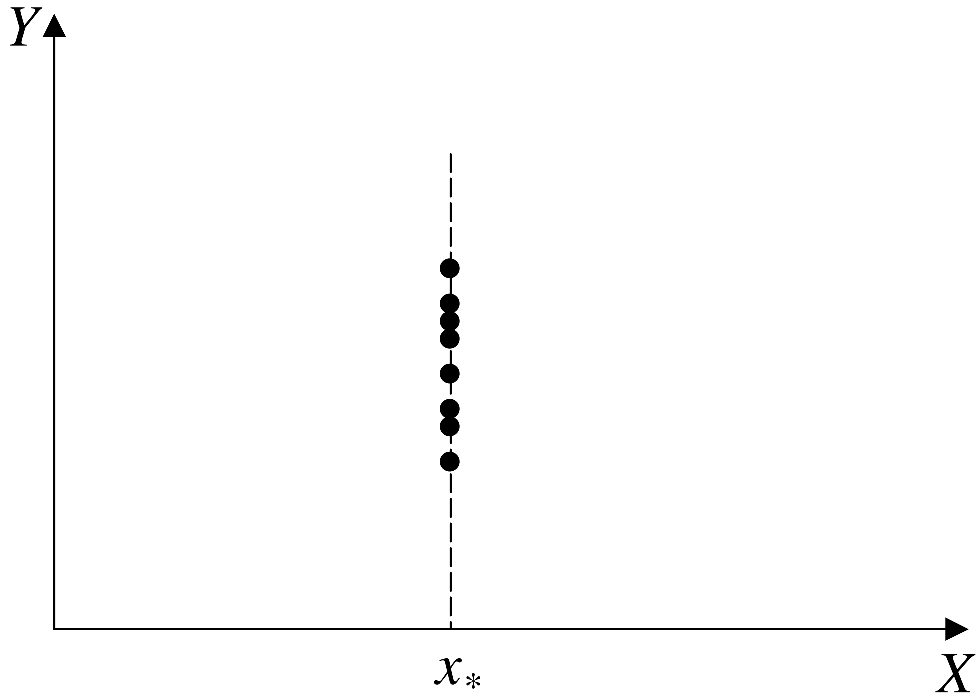
$$k = 1,\ \ p = 2 \qquad Y = \beta_0 + \beta_1 X + e$$

$$\left(\mathbf{X}^T\mathbf{X}\right)^{-1} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}^{-1}$$

$$= \frac{1}{n\sum x_i^2 - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \frac{1}{n\sum x_i^2 - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

$$= \frac{1}{n\sum x_i^2 - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 \sum Y_i - \sum x_i \sum x_i Y_i \\ n\sum x_i Y_i - \sum x_i \sum Y_i \end{bmatrix}$$

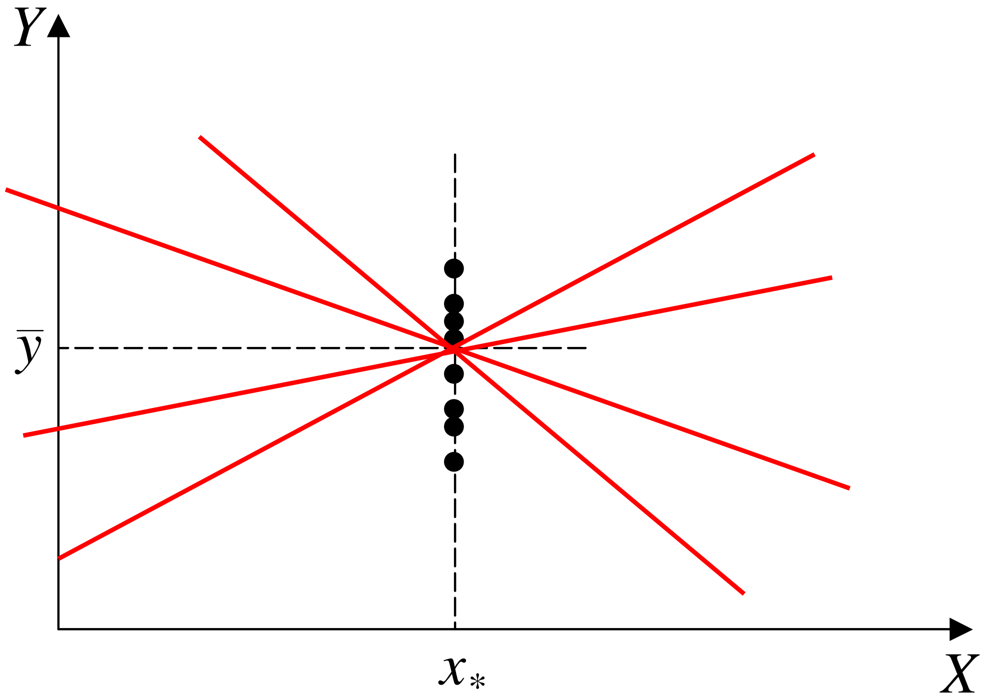


$$\sum_{i=1}^{n} e_i^{\,2} = \sum_{i=1}^{n} \left(Y_i - \beta_0 - \beta_1 x_i\right)^2$$

$$= \sum_{i=1}^{n} \left(Y_i - \overline{Y} + \overline{Y} - \beta_0 - \beta_1 \overline{x}\right)^2$$

$$= \sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2 + \sum_{i=1}^{n} \left(\overline{Y} - \beta_0 - \beta_1 \overline{x}\right)^2 \equiv \min \quad \text{se} \quad \overline{Y} = \beta_0 + \beta_1 \overline{x}$$

Vale per ogni retta $Y = \hat{\beta}_0 + \hat{\beta}_1 X$ passante per il punto $\left(\overline{x}, \overline{Y}\right)$



$$\sum_{i=1}^{n} e_i^{\,2} = \sum_{i=1}^{n} \left(Y_i - \beta_0 - \beta_1 x_i\right)^2$$

$$= \sum_{i=1}^{n} \left(Y_i - \overline{Y} + \overline{Y} - \beta_0 - \beta_1 \overline{x}\right)^2$$

$$= \sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2 + \sum_{i=1}^{n} \left(\overline{Y} - \beta_0 - \beta_1 \overline{x}\right)^2 \equiv \min \quad \text{se} \quad \overline{Y} = \beta_0 + \beta_1 \overline{x}$$

Vale per ogni retta $Y = \hat{\beta}_0 + \hat{\beta}_1 X$ passante per il punto $\left(\overline{x}, \overline{Y}\right)$

$$\mathbf{X} = \begin{bmatrix} 1 & x_* \\ 1 & x_* \\ \vdots & \vdots \\ 1 & x_* \end{bmatrix} \qquad \left(\mathbf{X}^T\mathbf{X}\right) = \begin{bmatrix} n & nx_* \\ nx_* & nx_*^2 \end{bmatrix}$$ Matrice *singolare*

$$\mathbf{X}^T\mathbf{Y} = \begin{bmatrix} \sum Y_i \\ \sum x_* Y_i \end{bmatrix} = \begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}$$

eq. normali $\begin{cases} n\hat{\beta}_0 + nx_*\hat{\beta}_1 = n\bar{Y} \\ nx_*\hat{\beta}_0 + nx_*^2\hat{\beta}_1 = nx_*\bar{Y} \end{cases} \qquad \Rightarrow n\hat{\beta}_0 + nx_*\hat{\beta}_1 = n\bar{Y}$

$$\hat{\beta}_1 = \frac{\bar{Y} - \hat{\beta}_0}{x_*} \qquad \hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \dfrac{\bar{Y} - \hat{\beta}_0}{x_*} \end{bmatrix}$$ *Infinite soluzioni*

Modello ANOVA (ad un fattore) $Y_{ij} = \mu + \tau_i + e_{ij}$ $\begin{cases} i = 1, \cdots, k \\ j = 1, \cdots, n_i \end{cases}$

Esempio:

$k = 2$ $\quad n_1 = 2 \quad n_2 = 3$

trattamenti	v.c. "risultati sperimentali"		
1	Y_{11}	Y_{12}	
2	Y_{21}	Y_{22}	Y_{23}

Modelli ANOVA $\Rightarrow$

$$Y_{11} = \mu + \tau_1 + e_{11}$$
$$Y_{12} = \mu + \tau_1 + e_{12}$$
$$Y_{21} = \mu + \tau_2 + e_{21}$$
$$Y_{22} = \mu + \tau_2 + e_{22}$$
$$Y_{23} = \mu + \tau_2 + e_{23}$$

$$2 \, \textit{variabili indicatrici} \text{ a valori } \begin{cases} 0 \\ 1 \end{cases} \Rightarrow \begin{array}{l} Y_{11} = \mu + \tau_1 1 + \tau_2 0 + e_{11} \\ Y_{12} = \mu + \tau_1 1 + \tau_2 0 + e_{12} \\ Y_{21} = \mu + \tau_1 0 + \tau_2 1 + e_{21} \\ Y_{22} = \mu + \tau_1 0 + \tau_2 1 + e_{22} \\ Y_{23} = \mu + \tau_1 0 + \tau_2 1 + e_{23} \end{array}$$

$$\boxed{Y_{ij} = \mu + \tau_1 x_{1ij} + \tau_2 x_{2ij} + e_{ij}} \qquad \begin{cases} i = 1,2 \\ j = 1, \cdots, n_i \end{cases} \quad \begin{cases} n_1 = 2 \\ n_2 = 3 \end{cases}$$

x_{1ij} è il valore di X_1 nel $j-esimo$ esperimento relativo al trattamento i (assenza/presenza di τ_1)

x_{2ij} è il valore di X_2 nel $j-esimo$ esperimento relativo al trattamento i (assenza/presenza di τ_2)

$$\Rightarrow \boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}}$$

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{21} \\ Y_{22} \\ Y_{23} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix} \qquad \mathbf{e} = \begin{pmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} \qquad \mathbf{X} = \begin{array}{c} \begin{array}{ccc} \mu & \tau_1 & \tau_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$r(\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = 2 < 3 = p \quad \Rightarrow \boxed{\text{non esiste } (\mathbf{X}^T\mathbf{X})^{-1}}$$

infinite soluzioni di $\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \mathbf{X}^T\mathbf{Y}$

$$\boxed{\text{SQ definite in modo unico}}$$

$$\text{In generale} \Rightarrow \quad \begin{cases} Y_{11} = \mu + \tau_1 1 + \tau_2 0 + \cdots + \tau_k 0 + e_{11} \\ \vdots \\ Y_{1n_1} = \mu + \tau_1 1 + \tau_2 0 + \cdots + \tau_k 0 + e_{1n_1} \end{cases}$$

$$\begin{cases} Y_{21} = \mu + \tau_1 0 + \tau_2 1 + \cdots + \tau_k 0 + e_{21} \\ \vdots \\ Y_{2n_2} = \mu + \tau_1 0 + \tau_2 1 + \cdots + \tau_k 0 + e_{2n_2} \end{cases}$$

$$\vdots$$

$$\begin{cases} Y_{k1} = \mu + \tau_1 0 + \tau_2 0 + \cdots + \tau_k 1 + e_{k1} \\ \vdots \\ Y_{kn_k} = \mu + \tau_1 0 + \tau_2 0 + \cdots + \tau_k 1 + e_{kn_k} \end{cases}$$

$$\boxed{Y_{ij} = \mu + \tau_1 x_{1ij} + \tau_2 x_{2ij} + \cdots + \tau_k x_{kij} + e_{ij}} \qquad \begin{cases} i = 1, \cdots, k \\ j = 1, \cdots, n_i \end{cases}$$

x_{tij} è il valore di X_t nel $j-esimo$ esperimento relativo al trattamento i

(assenza/presenza di τ_t, $t = 1, \cdots, k$)

$$\Rightarrow \quad \boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}}$$

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ \vdots \\ Y_{kn_k} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_k \end{pmatrix} \qquad \mathbf{e} = \begin{pmatrix} e_{11} \\ \vdots \\ e_{kn_k} \end{pmatrix} \qquad \mathbf{X} = \begin{array}{c} \begin{array}{ccccc} \mu & \tau_1 & \tau_2 & \cdots & \tau_k \end{array} \\ \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{1}_{n_2} & \mathbf{0} & \mathbf{1}_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{1}_{n_k} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_k} \end{pmatrix} \end{array}$$

$$p = k + 1$$

$$r(\mathbf{X}) = r\left(\mathbf{X}^T\mathbf{X}\right) = k < p \quad \Rightarrow \quad \text{infinite soluzioni di } \mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \mathbf{X}^T\mathbf{Y}$$

SQ definite in modo unico

Modelli di regressione $\Rightarrow$ Modelli lineari con $\mathbf{X}$ a *rango pieno*

Modelli ANOVA $\Rightarrow$ Modelli lineari con $\mathbf{X}$ *a rango ridotto*

Ipotesi sugli errori:

(1) $E(\mathbf{e}) = \mathbf{0} \quad \Rightarrow \quad E(\mathbf{Y}) = \mathbf{X\beta}$

(2 + 3) $V(\mathbf{e}) = E\left((\mathbf{e} - E(\mathbf{e}))(\mathbf{e} - E(\mathbf{e}))^T\right) = E\left(\mathbf{ee}^T\right) = \sigma^2\mathbf{I}$

$$\Rightarrow V(\mathbf{Y}) = E\left((\mathbf{Y} - \mathbf{X\beta})(\mathbf{Y} - \mathbf{X\beta})^T\right) = E\left(\mathbf{ee}^T\right) = \sigma^2\mathbf{I}$$

Proprietà dello stimatore di m.q. ($r = p$):

- $S(\hat{\boldsymbol{\beta}}) = (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})^T (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) = \min$

- $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \mathbf{M}\mathbf{Y}$ Stimatore lineare

- $E(\hat{\boldsymbol{\beta}}) = E\left((\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}\right) = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}$

  Stimatore non distorto

- $V(\hat{\boldsymbol{\beta}}) = V(\mathbf{M}\mathbf{Y}) = \mathbf{M}V(\mathbf{Y})\mathbf{M}^T$
  $= \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1} = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$

Teorema di Gauss-Markov

- Sia $\tilde{\boldsymbol{\beta}}$ uno stimatore lineare non distorto di $\boldsymbol{\beta}$

$$\Rightarrow \quad V\left(\mathbf{c}^T\tilde{\boldsymbol{\beta}}\right) \geq V\left(\mathbf{c}^T\hat{\boldsymbol{\beta}}\right) \quad \forall \mathbf{c}$$

Lo stimatore di m.q., e qualsiasi sua funzione lineare,
sono stimatori BLUE

Piano fattoriale con k fattori sperimentali (ciascuno a 2 livelli)

fattori	A_1	A_2	$\cdots$	A_k
livelli	a_{11} a_{12}	a_{21} a_{22}	$\cdots$	a_{k1} a_{k2}

$\boxed{A_i \text{ quantitativo}}$

$$x_{ij} = \frac{a_{ij} - \frac{1}{2}(a_{i1} + a_{i2})}{\frac{1}{2}(a_{i2} - a_{i1})} = \begin{cases} -1 & \text{per } j = 1 \\ +1 & \text{per } j = 2 \end{cases}$$

$\boxed{A_i \text{ qualitativo}}$

$$x_{ij} = \begin{cases} -1 & \text{per } A_i = a_{i1} \\ +1 & \text{per } A_i = a_{i2} \end{cases}$$

$\Rightarrow$ k fattori *quantitativi* $X_1, X_2, \cdots, X_k$ a livelli $\begin{cases} -1 \\ +1 \end{cases}$

Illustrazione per $k = 3$

dati	X_1	X_2	X_3
y_1	-1	-1	-1
y_2	1	-1	-1
y_3	-1	1	-1
y_4	1	1	-1
y_5	-1	-1	1
y_6	1	-1	1
y_7	-1	1	1
y_8	1	1	1

Obiettivo: valutare l'effetto dei fattori sperimentali

Modello di regressione (del primo ordine)

$$Y_i = E(Y_i) + e_i \qquad E(Y_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} \qquad i = 1, \cdots, 8$$

x_{1i}, x_{2i}, x_{3i} sono i livelli dei fattori X_1, X_2, X_3 per il trattamento i

In forma estesa:

$$\begin{aligned}
Y_1 &= \beta_0 + \beta_1(-1) + \beta_2(-1) + \beta_3(-1) + e_1 \\
Y_2 &= \beta_0 + \beta_1(+1) + \beta_2(-1) + \beta_3(-1) + e_2 \\
&\vdots \\
Y_8 &= \beta_0 + \beta_1(+1) + \beta_2(+1) + \beta_3(+1) + e_8
\end{aligned}$$

$$\Rightarrow \quad \boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}}$$

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_8 \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \qquad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_8 \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\boxed{\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{Y}}$$

$$\left(\mathbf{X}^T\mathbf{X}\right)=\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}=8I_4 \qquad \left(\mathbf{X}^T\mathbf{X}\right)^{-1}=\frac{1}{8}I_4$$

$$\mathbf{X}^T\mathbf{Y}=\begin{pmatrix} \sum Y_i \\ \sum x_{1i}Y_i \\ \sum x_{2i}Y_i \\ \sum x_{3i}Y_i \end{pmatrix} \qquad \hat{\boldsymbol{\beta}}=\begin{pmatrix} \frac{1}{8}\sum Y_i \\ \frac{1}{8}\sum x_{1i}Y_i \\ \frac{1}{8}\sum x_{2i}Y_i \\ \frac{1}{8}\sum x_{3i}Y_i \end{pmatrix}$$

$$V\left(\hat{\boldsymbol{\beta}}\right)=\sigma^2\left(\mathbf{X}^T\mathbf{X}\right)^{-1}$$

$$k=3 \quad \Rightarrow \qquad V\left(\hat{\boldsymbol{\beta}}\right)=\frac{\sigma^2}{8}I_4=\begin{pmatrix} \frac{\sigma^2}{8} & 0 & 0 & 0 \\ 0 & \frac{\sigma^2}{8} & 0 & 0 \\ 0 & 0 & \frac{\sigma^2}{8} & 0 \\ 0 & 0 & 0 & \frac{\sigma^2}{8} \end{pmatrix}$$

$$\Rightarrow \quad Cov\left(\hat{\beta}_j,\hat{\beta}_l\right)=0 \quad \forall\ j<l$$

Stime incorrelate (indipendenti) dei parametri (effetti)

$\Rightarrow$ Piano ortogonale

Inferenza sui parametri ($r = p$)

Modello stimato

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Y} = \mathbf{H}\mathbf{Y} \qquad \mathbf{H} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T$$

$\hat{Y}_i$: valori teorici di Y

$\mathbf{H} = \mathbf{H}^T$ Matrice simmetrica

$$\mathbf{H}^2 = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T = \mathbf{H}$$

Matrice idempotente

$\rightarrow$ Matrice proiezione

$\hat{e}_i = Y_i - \hat{Y}_i$: residui $\qquad \hat{\mathbf{e}} = \mathbf{Y} - \hat{\mathbf{Y}}$

$$\hat{\sigma}^2 = \frac{1}{n-p}\hat{\mathbf{e}}^T\hat{\mathbf{e}} = \frac{1}{n-p}\sum_i \left(Y_i - \hat{Y}\right)^2$$ stimatore non distorto di σ^2

Ipotesi: $\mathbf{e} \sim N_n\left[\mathbf{0}, \sigma^2 \mathbf{I}\right]$ $\qquad$ $\mathbf{Y} \sim N_n\left[\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}\right]$

$$\hat{\boldsymbol{\beta}} \sim N_p\left[\boldsymbol{\beta}, \sigma^2 \left(\mathbf{X}^T \mathbf{X}\right)^{-1}\right] \qquad \left(\mathbf{X}^T \mathbf{X}\right)^{-1} = \begin{bmatrix} s_{00} & s_{12} & \cdots & s_{1k} \\ & s_{11} & & s_{2k} \\ & & \ddots & \\ & & & s_{kk} \end{bmatrix}$$

$$\hat{\beta}_i \sim N\left[\beta_i, \sigma^2 s_{ii}\right] \hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}\left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{Y} = \mathbf{H}\mathbf{Y}$$

Intervallo di confidenza al livello $(1-\alpha)$ per β_i

limiti di confidenza $\Rightarrow$ $\qquad \hat{\beta}_i \mp t_{\alpha/2;n-p}\hat{\sigma}\sqrt{s_{ii}}$

Ipotesi statistiche: $H_0 : \beta_i = \beta_{i0} \quad \text{vs} \quad H_1 : \beta_i \neq \beta_{i0}$

Sotto $H_0 \Rightarrow$ $$t = \frac{\hat{\beta}_i - \beta_{i0}}{\hat{\sigma}\sqrt{s_{ii}}} \sim t_{n-p}$$ *statistica-test*

$$t_0 = \frac{\hat{\beta}_i - \beta_{i0}}{\hat{\sigma}\sqrt{s_{ii}}} \sim t_{n-p}$$ *valore osservato*

Si rifiuta H_0 (al livello di significatività α) se $\begin{cases} t_0 > t_{\alpha/2;n-p} \\ t_0 < -t_{\alpha/2;n-p} \end{cases}$

Ipotesi tipiche: $H_0 : \beta_i = 0 \quad \text{vs} \quad H_1 : \beta_i \neq 0$

Previsione

Modello stimato $\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$

$$\mathbf{x}_* = \begin{pmatrix} 1 \\ x_{1*} \\ x_{2*} \\ \\ x_{k*} \end{pmatrix}$$

Previsore puntuale

$$\hat{Y}_* = \mathbf{x}_*^T \hat{\boldsymbol{\beta}} = \hat{\beta}_0 + \hat{\beta}_1 x_{1*} + \hat{\beta}_2 x_{2*} \cdots + \hat{\beta}_k x_{k*}$$

$$E\left(\hat{Y}_*\right) = \mathbf{x}_*^T \boldsymbol{\beta}$$

Stimatore non distorto

$$V\left(\hat{Y}_*\right) = \mathbf{x}_*^T V\left(\hat{\boldsymbol{\beta}}\right) \mathbf{x}_* = \sigma^2 \mathbf{x}_*^T \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{x}_*$$

$$\hat{Y}_* \sim N\left[\mathbf{x}_*^T \boldsymbol{\beta}, \sigma^2 \mathbf{x}_*^T \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{x}_*\right]$$

$$\frac{\hat{Y}_* - \mathbf{x}_*^T \boldsymbol{\beta}}{\hat{\sigma} \sqrt{\mathbf{x}_*^T \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{x}_*}} \sim t_{n-p}$$

Intervallo di confidenza al livello $(1-\alpha)$ per $\mathbf{x}_*^T \boldsymbol{\beta}$

limiti di confidenza $\Rightarrow \quad \hat{y}_* \mp t_{\alpha/2;n-p} \hat{\sigma} \sqrt{\mathbf{x}_*^T \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{x}_*}$

$k=1, \;\; p=2 \qquad Y = \beta_0 + \beta_1 X + e \qquad \mathbf{x}_* = \begin{pmatrix} 1 \\ x_* \end{pmatrix}$

$V\left(\hat{Y}_*\right) = \dfrac{\sigma^2}{n}\left[1 + \dfrac{(x_* - \bar{x})^2}{\sigma_X^2}\right] \qquad$ Se $x_* = \bar{x} \;\; \Rightarrow \;\; V\left(\hat{Y}_*\right) = \dfrac{\sigma^2}{n} = \min$

$I.C.$ per $\mathbf{x}_*^T \boldsymbol{\beta} \;\; \Rightarrow$