$$r(\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = r \leq p$$

$\mathbf{G}$ è *una* inversa generalizzata di $\mathbf{X}^T\mathbf{X}$ se : $(\mathbf{X}^T\mathbf{X})\mathbf{G}(\mathbf{X}^T\mathbf{X}) = \mathbf{X}^T\mathbf{X}$

<u>Proprietà di $\mathbf{G}$:</u>

- $\mathbf{H} = \mathbf{X}\mathbf{G}\mathbf{X}^T$ è invariante rispetto a $\mathbf{G}$
- $\mathbf{H}\mathbf{X} = \mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{X}$ e $\mathbf{X}^T\mathbf{H} = \mathbf{X}^T\mathbf{X}\mathbf{G}\mathbf{X}^T = \mathbf{X}^T$
- $\mathbf{H}$ è una proiezione

$\mathbf{H} = \mathbf{H}^T$ che $\mathbf{G}$ sia simmetrica o meno

$\mathbf{S}$ i.g. simmetrica di $\mathbf{X}\mathbf{X}^T \Rightarrow \mathbf{X}\mathbf{S}\mathbf{X}^T$ simmetrica
ma $\mathbf{X}\mathbf{S}\mathbf{X}^T = \mathbf{X}\mathbf{G}\mathbf{X}^T \Rightarrow \mathbf{X}\mathbf{G}\mathbf{X}^T$ simmetrica

$$\mathbf{H}^2 = \mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X}\mathbf{G}\mathbf{X}^T = \mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{H} = \mathbf{H}$$

- $r(\mathbf{H}) = tr(\mathbf{H}) = r$

$$r(\mathbf{H}) = r(\mathbf{X}\mathbf{G}\mathbf{X}^T) = tr(\mathbf{X}\mathbf{G}\mathbf{X}^T) = tr(\mathbf{G}\mathbf{X}^T\mathbf{X}) = r(\mathbf{G}\mathbf{X}^T\mathbf{X}) \leq r(\mathbf{X}^T\mathbf{X}) = r$$

$$r(\mathbf{H}) = r(\mathbf{X}\mathbf{G}\mathbf{X}^T) = r(\mathbf{G}\mathbf{X}^T\mathbf{X}) \geq r(\mathbf{X}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = r$$

$$\Rightarrow \quad r(\mathbf{H}) = r$$

- $\mathbf{X} = (\underset{n\times r}{\tilde{\mathbf{X}}}, \mathbf{X}_0), \quad \tilde{\mathbf{X}}: \ r(\tilde{\mathbf{X}}) = r(\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = r$

  $\Rightarrow$ *una* i.g. di $\mathbf{X}^T\mathbf{X}$ è $\quad \mathbf{G} = \begin{pmatrix} (\tilde{\mathbf{X}}^T\tilde{\mathbf{X}})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$

Se $r(\mathbf{X}) = r(\mathbf{X}^T\mathbf{X}) = p \quad \Rightarrow \quad \mathbf{G} = (\mathbf{X}^T\mathbf{X})^{-1}$ è unica

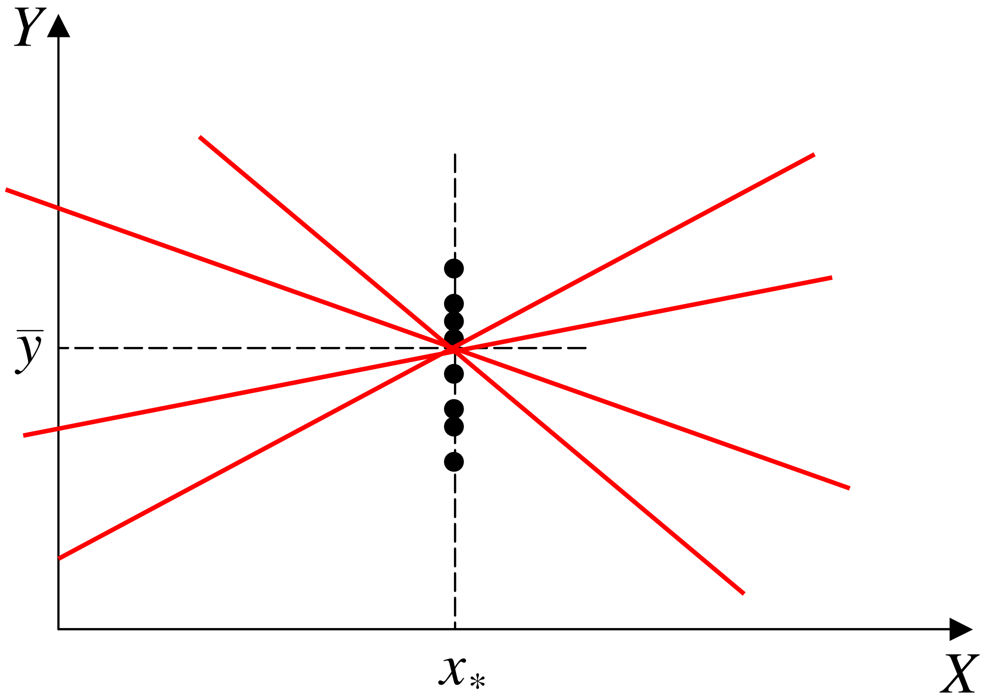
$$\mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{Y}$$ equazioni *normali*

$$\hat{\boldsymbol{\beta}} = \mathbf{G}\mathbf{X}^T\mathbf{Y}$$ è *una* soluzione (di minimi quadrati)

infatti $\mathbf{X}^T\mathbf{X}\left(\mathbf{G}\mathbf{X}^T\mathbf{Y}\right) = \mathbf{X}^T\mathbf{Y}$

Se $r = p \quad \Rightarrow \quad \mathbf{G} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}$

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Y}$$ unica soluzione



$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \dfrac{\bar{Y} - \hat{\beta}_0}{x_*} \end{bmatrix}$$

$$\left(\mathbf{X}^T\mathbf{X}\right)\hat{\boldsymbol{\beta}} = \mathbf{X}^T\mathbf{Y} \;\Rightarrow\; \begin{bmatrix} n & nx_* \\ nx_* & nx_*^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}$$

$$\mathbf{G} \text{ i.g. di } \mathbf{X}^T\mathbf{X} \;\Rightarrow\; \hat{\boldsymbol{\beta}} = \mathbf{G}\mathbf{X}^T\mathbf{Y}$$

$$(1)\quad \mathbf{G}=\begin{bmatrix} n^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{\beta}}=\begin{bmatrix} n^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}=\begin{bmatrix} \bar{Y} \\ 0 \end{bmatrix}$$

$$(2)\quad \mathbf{G}=\begin{bmatrix} 0 & 0 \\ 0 & \left(nx_*^2\right)^{-1} \end{bmatrix} \Rightarrow \hat{\boldsymbol{\beta}}=\begin{bmatrix} 0 & 0 \\ 0 & \left(nx_*^2\right)^{-1} \end{bmatrix}\begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}=\begin{bmatrix} 0 \\ \bar{Y}/x_* \end{bmatrix}$$

$$(3)\quad \mathbf{G}=\begin{bmatrix} 0 & \left(nx_*\right)^{-1} \\ 0 & 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{\beta}}=\begin{bmatrix} 0 & \left(nx_*\right)^{-1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}=\begin{bmatrix} \bar{Y} \\ 0 \end{bmatrix}$$

$$(4)\quad \mathbf{G}=\begin{bmatrix} 0 & 0 \\ \left(nx_*\right)^{-1} & 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{\beta}}=\begin{bmatrix} 0 & 0 \\ \left(nx_*\right)^{-1} & 0 \end{bmatrix}\begin{bmatrix} n\bar{Y} \\ nx_*\bar{Y} \end{bmatrix}=\begin{bmatrix} 0 \\ \bar{Y}/x_* \end{bmatrix}$$

Alternative (pratiche) all'uso dell'inversa generalizzata:

- modifica del modello per rendere $\mathbf{X}^T\mathbf{X}$ non singolare
- acquisizione di nuovi dati per rendere $\mathbf{X}^T\mathbf{X}$ non singolare
- restrizioni lineari per rendere $\mathbf{X}^T\mathbf{X}$ non singolare

  Es.: si pongono pari a zero tutti i parametri associati
  alle colonne di $\mathbf{X}$ che generano dipendenza

- restrizioni non lineari e soluzione del problema di minimi quadrati

  Es. *ridge regression*

Proprietà dello stimatore di m.q. ($r \le p$):

- $S\left(\hat{\boldsymbol{\beta}}\right) = \left(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right)^T \left(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right) = \min$

- $\hat{\boldsymbol{\beta}} = \mathbf{G}\mathbf{X}^T\mathbf{Y} = \mathbf{M}\mathbf{Y}$ Stimatore lineare

- $E\left(\hat{\boldsymbol{\beta}}\right) = E\left(\mathbf{G}\mathbf{X}^T\mathbf{Y}\right) = \mathbf{G}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$ stimatore distorto

  se $r = p$ $\quad E\left(\hat{\boldsymbol{\beta}}\right) = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}$ non distorto

- $V\left(\hat{\boldsymbol{\beta}}\right) = V\left(\mathbf{M}\mathbf{Y}\right) = \mathbf{M}V\left(\mathbf{Y}\right)\mathbf{M}^T = \sigma^2\mathbf{G}\mathbf{X}^T\mathbf{X}\mathbf{G}^T$ non unica

  se $r = p$ $\quad V\left(\hat{\boldsymbol{\beta}}\right) = \sigma^2\left(\mathbf{X}^T\mathbf{X}\right)^{-1}$

Funzioni stimabili

$$\hat{\boldsymbol{\beta}} \text{ non è unico } \Rightarrow \boldsymbol{\beta} \text{ non è } \textit{stimabile}$$

$$\mathbf{c}^T\boldsymbol{\beta} \text{ è una funzione stimabile se } \mathbf{c}^T = \mathbf{d}^T\mathbf{X}$$

$$\Rightarrow \mathbf{c}^T\hat{\boldsymbol{\beta}} \text{ stimatore m.q. di } \mathbf{c}^T\boldsymbol{\beta}$$

Proprietà:

- $\mathbf{c}^T\hat{\boldsymbol{\beta}} = \mathbf{d}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{Y}$ invariante rispetto a $\mathbf{G}$
- $E\left(\mathbf{c}^T\hat{\boldsymbol{\beta}}\right) = \mathbf{d}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \mathbf{d}^T\mathbf{X}\boldsymbol{\beta} = \mathbf{c}^T\boldsymbol{\beta}$ non distorto
- $V\left(\mathbf{c}^T\hat{\boldsymbol{\beta}}\right) = \sigma^2\mathbf{d}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{d} = \sigma^2\mathbf{d}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{d} = \sigma^2\mathbf{c}^T\mathbf{G}\mathbf{c}$ unica

Criterio di stimabilità $\Rightarrow \quad \mathbf{c}^T\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{c}^T$

$$\mathbf{c}^T\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{c}^T \quad \Rightarrow \quad \mathbf{c}^T = \mathbf{c}^T\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{d}^T\mathbf{X}$$

infatti

$$\mathbf{c}^T = \mathbf{d}^T\mathbf{X} \quad \Rightarrow \quad \mathbf{c}^T\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{d}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{X} = \mathbf{d}^T\mathbf{X} = \mathbf{c}^T$$

se $r = p$ $\quad \boldsymbol{\beta}$ e $\mathbf{c}^T\boldsymbol{\beta}$, $\mathbf{c}$ qualsiasi, sono stimabili

Teorema di Gauss-Markov

Sia $\mathbf{c}^T\tilde{\boldsymbol{\beta}}$ uno stimatore lineare non distorto di $\mathbf{c}^T\boldsymbol{\beta}$

$$\Rightarrow \quad V\left(\mathbf{c}^T\tilde{\boldsymbol{\beta}}\right) \geq V\left(\mathbf{c}^T\hat{\boldsymbol{\beta}}\right) \qquad \mathbf{c}^T\hat{\boldsymbol{\beta}} \text{ è BLUE}$$

se $r = p$ $\quad \hat{\boldsymbol{\beta}}$ e $\mathbf{c}^T\hat{\boldsymbol{\beta}}$, $\mathbf{c}$ qualsiasi, sono BLUE

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{Y} = \mathbf{H}\mathbf{Y} \quad \text{invariante rispetto a } \mathbf{G}$$

$$\text{se } r = p \quad \mathbf{H} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T$$

$$\mathbf{1}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{1}^T\mathbf{Y} \;\Rightarrow\; \bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x}_1 + \hat{\beta}_2\bar{x}_2 + \cdots + \hat{\beta}_k\bar{x}_k$$

$$\mathbf{1}^T\hat{\mathbf{Y}} = \mathbf{1}^T\mathbf{Y} \;\Rightarrow\; \sum \hat{y}_i = \sum y_i$$

$$\hat{\mathbf{Y}}^T\hat{\mathbf{Y}} = \mathbf{Y}^T\mathbf{H}^T\mathbf{H}\mathbf{Y} = \mathbf{Y}^T\mathbf{H}\mathbf{Y}$$

$$\hat{\mathbf{Y}}^T\mathbf{Y} = \mathbf{Y}^T\hat{\mathbf{Y}} = \mathbf{Y}^T\mathbf{H}\mathbf{Y}$$

$$\Rightarrow \hat{\mathbf{Y}}^T\hat{\mathbf{Y}} = \hat{\mathbf{Y}}^T\mathbf{Y} = \mathbf{Y}^T\hat{\mathbf{Y}}$$

$$n\bar{Y}^2 = n\left(\frac{1}{n}\sum Y_i\right)^2 = \frac{1}{n}\left(\mathbf{Y}^T\mathbf{1}\right)^2 = \frac{1}{n}\mathbf{Y}^T\mathbf{1}\mathbf{1}^T\mathbf{Y} = \mathbf{Y}^T\left(\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{Y}$$

$$SQT = \sum \left(Y_i - \overline{Y}\right)^2 = \sum Y_i^2 - n\overline{Y}^2 =$$
$$= \mathbf{Y}^T\mathbf{Y} - \mathbf{Y}^T\left(\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{Y} = \mathbf{Y}^T\left(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{Y}$$

$$SQM = \sum \left(\hat{Y}_i - \overline{Y}\right)^2 = \sum \hat{Y}_i^2 - n\overline{Y}^2 =$$
$$= \mathbf{Y}^T\mathbf{H}\mathbf{Y} - \mathbf{Y}^T\left(\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{Y} = \mathbf{Y}^T\left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{Y}$$

$$SQR = \sum \left(Y_i - \hat{Y}\right)^2 = \left(\mathbf{Y} - \hat{\mathbf{Y}}\right)^T\left(\mathbf{Y} - \hat{\mathbf{Y}}\right) =$$
$$= \mathbf{Y}^T\mathbf{Y} - \mathbf{Y}^T\hat{\mathbf{Y}} - \hat{\mathbf{Y}}^T\mathbf{Y} + \hat{\mathbf{Y}}^T\hat{\mathbf{Y}} = \mathbf{Y}^T\mathbf{Y} - \hat{\mathbf{Y}}^T\hat{\mathbf{Y}} = \mathbf{Y}^T\left(\mathbf{I} - \mathbf{H}\right)\mathbf{Y}$$

$$SQT = SQM + SQR$$

$$0 \le R^2 = \frac{SQM}{SQT} \le 1 \quad \text{coeff. di determinazione}$$

Teoremi sulla forme quadratiche

$$\mathbf{Z} \sim N_n[\boldsymbol{\mu}, \mathbf{V}]$$

(1) $$\mathbf{Z}^T\mathbf{A}\mathbf{Z} \sim \chi^{2'}\left[r(\mathbf{A}); \frac{1}{2}\boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}\right] \iff \mathbf{AV} \text{ è idempotente}$$

$$\Rightarrow E(\mathbf{Z}^T\mathbf{A}\mathbf{Z}) = tr(\mathbf{AV}) + \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}$$

(2) $\mathbf{Z}^T\mathbf{A}\mathbf{Z}$ e $\mathbf{Z}^T\mathbf{B}\mathbf{Z}$ sono indipendenti $\iff$ $\mathbf{AVB} = \mathbf{0}$

(3) Siano $U_1 \sim \chi^{2'}[n_1; \lambda]$ e $U_2 \sim \chi^2[n_2]$ v.c. indipendenti

$$\Rightarrow \frac{U_1/n_1}{U_2/n_2} \sim F'[n_1, n_2; \lambda]$$

$$\mathbf{Y} \sim N_n\left[\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}\right] \;\Rightarrow\; \frac{1}{\sigma}\mathbf{Y} \sim N_n\left[\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I}\right]$$

$$\frac{SQM}{\sigma^2} = \left(\frac{1}{\sigma}\mathbf{Y}\right)^T \left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\left(\frac{1}{\sigma}\mathbf{Y}\right)$$

$$\left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = \mathbf{H} - \frac{1}{n}\mathbf{H}\mathbf{1}\mathbf{1}^T - \frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{H} + \frac{1}{n}\mathbf{1}\mathbf{1}^T = \left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)$$

idempotente

Oss.: $\mathbf{1}^T\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{1}^T\mathbf{X}\mathbf{G}\mathbf{X}^T\mathbf{Y} = \mathbf{1}^T\mathbf{H}\mathbf{Y} = \mathbf{1}^T\mathbf{Y} \;\Rightarrow\; \mathbf{1}^T\mathbf{H} = \mathbf{1}^T$

$$r\left(\mathbf{H} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = tr(\mathbf{H}) - tr\left(\frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = r - 1$$

$$\text{par. di non centr.} \Rightarrow \frac{1}{2\sigma^2}\boldsymbol{\beta}^T\mathbf{X}^T\left(\mathbf{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{X}\boldsymbol{\beta}$$

$$\boldsymbol{\beta}^T\mathbf{X}^T\left(\mathbf{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{X}\boldsymbol{\beta}=\boldsymbol{\beta}^T\mathbf{X}^T\left(\mathbf{I}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{X}\boldsymbol{\beta}$$

$$\text{poiché} \quad \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{H}\mathbf{X}\boldsymbol{\beta}=\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

$$\boldsymbol{\beta}=\begin{pmatrix}\beta_0\\ \boldsymbol{\beta}_1\end{pmatrix} \;;\quad \mathbf{X}=\left(\mathbf{1},\mathbf{X}_1\right) \;\Rightarrow\; \mathbf{X}\boldsymbol{\beta}=\mathbf{1}\beta_0+\mathbf{X}_1\boldsymbol{\beta}_1$$

$$\boldsymbol{\beta}^T \mathbf{X}^T \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right) \mathbf{X}\boldsymbol{\beta} = \left(\mathbf{1}\beta_0 + \mathbf{X}_1 \boldsymbol{\beta}_1 \right)^T \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right) \left(\mathbf{1}\beta_0 + \mathbf{X}_1 \boldsymbol{\beta}_1 \right)$$

$$= \left(\beta_0 \mathbf{1}^T + \boldsymbol{\beta}_1^T \mathbf{X}_1^T \right) \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right) \left(\mathbf{1}\beta_0 + \mathbf{X}_1 \boldsymbol{\beta}_1 \right)$$

$$= \left(\beta_0 \mathbf{1}^T + \boldsymbol{\beta}_1^T \mathbf{X}_1^T - \frac{1}{n} \beta_0 n \mathbf{1}^T - \frac{1}{n} \boldsymbol{\beta}_1^T \mathbf{X}_1^T \mathbf{1}\mathbf{1}^T \right) \left(\mathbf{1}\beta_0 + \mathbf{X}_1 \boldsymbol{\beta}_1 \right)$$

$$= \beta_0 \boldsymbol{\beta}_1^T \mathbf{X}_1^T \mathbf{1} + \boldsymbol{\beta}_1^T \mathbf{X}_1^T \mathbf{X}_1 \boldsymbol{\beta}_1 - \frac{1}{n} \beta_0 \boldsymbol{\beta}_1^T \mathbf{X}_1^T \mathbf{1} n - \boldsymbol{\beta}_1^T \mathbf{X}_1^T \frac{1}{n} \mathbf{1}\mathbf{1}^T \mathbf{X}_1 \boldsymbol{\beta}_1$$

$$= \boldsymbol{\beta}_1^T \mathbf{X}_1^T \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right) \mathbf{X}_1 \boldsymbol{\beta}_1$$

$$\text{teorema (1)} \Rightarrow \frac{SQM}{\sigma^2} \sim \chi^{2'} \left[r - 1 ; \frac{1}{2\sigma^2} \boldsymbol{\beta}_1^T \mathbf{X}_1^T \left(\mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right) \mathbf{X}_1 \boldsymbol{\beta}_1 \right]$$

$$\frac{SQR}{\sigma^2} = \left(\frac{1}{\sigma}\mathbf{Y}\right)^T (\mathbf{I}-\mathbf{H})\left(\frac{1}{\sigma}\mathbf{Y}\right)$$

$$(\mathbf{I}-\mathbf{H})(\mathbf{I}-\mathbf{H}) = \mathbf{I}-\mathbf{H}-\mathbf{H}+\mathbf{H} = \mathbf{I}-\mathbf{H}$$

$$r(\mathbf{I}-\mathbf{H}) = tr(\mathbf{I}) - tr(\mathbf{H}) = n-r$$

$$\boldsymbol{\beta}^T\mathbf{X}^T(\mathbf{I}-\mathbf{H})\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{H}\mathbf{X}\boldsymbol{\beta} = 0$$

teorema (1) $\Rightarrow \dfrac{SQR}{\sigma^2} \sim \chi^2[n-r]$

$$E\left(\frac{SQR}{\sigma^2}\right) = n-r \quad \Rightarrow \quad E\left(\hat{\sigma}^2\right) = E\left(\frac{SQR}{n-r}\right) = \sigma^2$$

$$\left(\mathbf{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T\right)(\mathbf{I}-\mathbf{H})=\mathbf{H}-\mathbf{H}-\frac{1}{n}\mathbf{1}\mathbf{1}^T+\frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{H}=\mathbf{0}$$

teorema (2) $\Rightarrow \dfrac{SQM}{\sigma^2}$ e $\dfrac{SQR}{\sigma^2}$ sono v.c. indipendenti

$$\Rightarrow \quad f=\frac{\dfrac{SQM}{\sigma^2}\Big/ r-1}{\dfrac{SQR}{\sigma^2}\Big/ n-r}=\frac{SQM\,/\,r-1}{\hat{\sigma}^2}$$

teorema (3) $\Rightarrow f \sim F'\left[r-1, n-r; \dfrac{1}{2\sigma^2}\boldsymbol{\beta}_1^T\mathbf{X}_1^T\left(\mathbf{I}-\dfrac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mathbf{X}_1\boldsymbol{\beta}_1\right]$

Tabella di analisi della varianza

Fonte di variabilità	SQ	g.l.	MQ	Rapporto f
Modello	SQM	r-1	SQM/(r-1)	f_0
Residua	SQR	n-r	SQR/(n-r)	
Totale	SQT	n-1		

$$\text{Sotto } H_0 : \mathbf{X}_1\boldsymbol{\beta}_1 = \mathbf{0} \;\Rightarrow\; f \sim F[r-1, n-r]$$

$$p-value \quad p = P[f > f_0] \;\Rightarrow\; \text{si rifiuta } H_0 \text{ per ogni } \alpha > p$$

$$\text{ANOVA (ad un fattore)} \quad H_0 : \mathbf{X}_1\boldsymbol{\beta}_1 = \mathbf{0} \;\Rightarrow\; \tau_1 = \tau_2 = \cdots = \tau_k = 0$$

$$\text{se } r = p \quad H_0 : \mathbf{X}_1\boldsymbol{\beta}_1 = \mathbf{0} \;\Rightarrow\; \beta_1 = \beta_2 = \cdots = \beta_k = 0$$