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TOPIC 3: RISK CONTROL - Ruin Theory, Solvency.

A RISK THEORETICAL MODEL FOR ASSESSING THE SOLVENCY PROFILE OF A GENERAL INSURER

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Abstract

A risk theoretical simulation model is here applied in order to assess the default risk of a general insurer along a medium-term time horizon. Different ruin barriers are regarded and by the results of the simulation model is then built up a Risk vs Return trade-off to analyse the most appropriate strategies in order to satisfy the insurer targets.
Clearly not only profitability level but also risk measures must be taken into account, with special reference to minimum capital levels required by the insurance regulators. At this regard different suitable strategies may be pursued and, among these, reinsurance is one of the most relevant for the insurance risk management. Only conventional covers as quota share and excess of loss reinsurance are here regarded, but it is emphasized how effective they can be on the risk/return profile of a general insurer.
To increase the volume of business is a natural target for the management of an insurance company, but that may cause a need of either new capital or/and reinsurance for solvency requirements and consequently a reduction in profitability for stockholders is likely to occur.

Keywords: Non-Life Insurance Solvency, Risk Theory, Monte Carlo simulations, Return on Equity, Probability of Ruin, Minimum Solvency Margin.

1. INTRODUCTION.

Many studies have been carried out on the topic of insurance solvency and extensive researches have been appointed by governments and various institutions over the last decades. Among these, a particular mention has to be deserved to the

well known studies carried out by Campagne, Buol and De Mori for both life and non-life insurance solvency, on whose results the minimum solvency margin in the EEC countries were established in 70's for both life and non-life insurers. The results of those studies are still a relevant benchmark also in the most recent European and North American actuarial studies, analysing the Risk-Based Capital system applied in USA and the reform of the EU minimum solvency margin formula[1]. Notwithstanding the numerous and relevant criticism addressed to their studies it is to be recognised to them the merit to have fixed, a long time ago nowdays, a first general criteria for the solvency conditions and to have promoted a larger cooperation on the matter amongst the European countries.
Anyway, notwithstanding in the assurance legislation a simple formula for the minimum solvency margin is needed, a universal formula is commonly considered to be an impossible achievement, moreover for the increasing complexity of the real insurance world. At this regard, many researches[2] have pointed out how the simulation of comprehensive model may represent a suitable tool for the supervisory authority, in order to perform, after the "solvency test" (that may be regarded as a tool of "first level control"), a "second level control" taking into account all possible features of the company which can not be simply considered in the "first level" analysis.

These studies have mainly made use of simulation techniques in order to be able to draw some conclusions for whatever insurer. In the present paper, it is emphasized how such kind of models may be suitable for the risk management in general insurance, with particular reference to underwriting, pricing, reserving, reinsurance and investment. The attention is here focused only on the pure underwriting risk with the analysis of the reinsurance impact, modelling a single-line general insurer with a portfolio affected by short-term fluctuations on claim frequencies but without any claim reserving run-off.

When a solvency analysis is carried out, great attention must be paid to the well-known trade-off in force in insurance (Solvency vs Profitability) affecting a large part of the management strategies.
Indeed, the main pillars of the insurance management are:

- high growth in the volume of business and in the market share;
- sound financial strength;
- competitive return for stockholders' capital.

To increase the volume of business is a natural target for the management, but that may cause a need of new capital for solvency requirements and consequently a reduction in profitability of equity is likely to occur.

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[1] See e.g. Report O.C.S.E. (1961), Actuarial Advisory Committee to the NAIC Property & Casualty Risk-Based Capital Working Group (1992), Johnsen et al. (1993), Müller Working Party (1997).

[2] As to pioneer researches in general insurance at this regard, see e.g. Pentikäinen and Rantala (1982), British General Insurance Solvency Group (1987), Pentikäinen et al. (1989) and Daykin and Hey (1990).

In other words, the main goal for the insurance management is how to increase return for stockholders with the relevant constraint to afford all underwritten liabilities and to guarantee them with a relevant risk capital invested into the company such to fulfil minimum capital requirements approved by the supervisory authority, and possible extra voluntary risk capital to face supplementary insurance risks.
An appropriate risk management analysis is then needed in order to assess the measure of risk (probability of ruin, capital-at-risk, unconditional expected shortfall, etc.), clearly depending on both the structure of its insurance and investment portfolio and the risk capital available at the moment of the evaluation. Once the tolerable ruin probability[3] is fixed and regarded as suitable for the company, that is the upper limit to be not exceeded and then for a short-medium term an estimate of the actual probability of ruin is needed together with the probability distribution of the return on equity linked to alternative strategies.

2. A RISK THEORETICAL APPROACH FOR MODELLING THE RISK RESERVE OF A GENERAL INSURER.

The main target of the present paper is to analyse the risk profile of a general insurer specialized in a single personal line of casualty, on both solvency and return benchmarks, and moreover to show the effects of some traditional reinsurance treaties. The framework of the model provides a risk theoretical approach where the underwriting risk is almost exclusively dealt with, and at the present stage of the model the financial variables are simply regarded as deterministic and the run-off risk rising from loss reserving is not considered.
In classical Risk-Theory literature the stochastic Risk Reserve $\tilde{U}_t$ at the end of the generic year t is given by:

$$(1) \qquad \tilde{U}_t = (1+j)\cdot\tilde{U}_{t-1} + \left[(B_t - \tilde{X}_t - E_t) - (B_t^{RE} - \tilde{X}_t^{RE} - C_t^{RE})\right]\cdot(1+j)^{1/2}$$

with gross premiums volume (B_t), stochastic aggregate claims amount ($\tilde{X}_t$) and general and acquisition expenses (E_t) realized in the middle of the year, whereas j is the annual rate of investment return, assumed to be a constant risk-free rate. As to reinsurance, B_t^{RE} denotes the gross premiums volume ceded to reinsurer

[3] As emphasized in Coutts and Thomas (1997) "*the risk tolerance level of an individual company is clearly a matter for its Board of Director to establish, subject to regulatory minimum standards*", and the concept of probability of ruin may be used as a measure of this "*risk tolerance*". In particular, these authors defined five different measures of ruin, according the failure of management target, the regulatory intervention level, net worth turning negative, exhaustion of cash and investments and, finally, inability to dispose of illiquid investments.

whereas $\widetilde{X}_t^{RE}$ and C_t^{RE} are respectively the amount of claims refunded by reinsurer and the reinsurance commissions. Neither dividends nor taxation are considered into the model.

The gross premium amount is composed of risk premium P_t=E ($\widetilde{X}_t$), safety loadings applied as a (constant) quota of the risk premium $\lambda * P_t$ and of the expenses loading as a (constant) coefficient c applied on the gross premium:

$$B_t = P_t + \lambda \cdot P_t + c \cdot B_t$$

Notwithstanding the risk loading coefficient λ is kept constant over the full time horizon, it is initially computed according to the standard deviation premium principle for the initial portfolio structure as follows:

$$\lambda \cdot E(\widetilde{X}) = \beta \cdot \sigma(\widetilde{X})$$

in order to ask for a risk loading amount equal to β for each unit of the standard deviation of the total amount of claims.

Disregarding reinsurance covers, in case the actual expenses are equal to the expense loadings ($E_t = c \cdot B_t$), the classical risk reserve equation (1) becomes:

(1bis) $$\widetilde{U}_t = (1+j) \cdot \widetilde{U}_{t-1} + \left[(1+\lambda) \cdot P_t - \widetilde{X}_t\right] \cdot (1+j)^{1/2}$$

It is here assumed that claims settlement will take place in the same year as the claim event and therefore no claims provision at the end of the year is needed. Actually, for many general insurance lines (e.g. third-party liability) the run-off risk concerning the development of the initial estimate of claim reserve is not negligible at all and therefore is an additive source of risk, but on the other hand, at the present stage of the model it may be assumed to be offset by the investment returns of the claim reserves, not accounted for in the formula (1bis).

The nominal gross premium volume increases yearly by the claim inflation rate (i) and the real growth rate (g):

$$B_t = (1+i) \cdot (1+g) \cdot B_{t-1}$$

assumed rates i and g to be constant in the regarded time horizon.

Following the collective approach, the aggregate claims amount $\widetilde{X}_t$ is given by a compound process:

(2) $$\widetilde{X}_t = \sum_{i=1}^{\widetilde{k}_t} \widetilde{Z}_{i,t}$$

where $\widetilde{k}_t$ is the random variable of the number of claims occurred in the year t and $\widetilde{Z}_{i,t}$ the random claim size of the i-th claim occurred at year t.
As well known an usual assumption in general insurance for the number of claims distribution is the Poisson law, and having assumed a dynamic portfolio the Poisson parameter will be increasing (or decreasing) recursively year by year by the real growth rate g. It means that $\widetilde{k}_t$ is Poisson distributed with parameter $n_t = n_0 \cdot (1+g)^t$ depending on the time.

In practice the simple Poisson law frequently fails to provide a satisfactory representation of the actual claim number distribution. Usually the number of claims is affected by other types of fluctuations[4] than pure random fluctuations:

a) Trends: when a slow moving change of the claim probabilities is occurring. They can produce an either increase or decrease of the expected value since a systematic change in the line environment conditions;
b) Short-period fluctuations: when fluctuations are affecting only in the short-term (usually less than a year) the assumed probability distribution, without any time-dependency. In practice, they can be caused for instance by meteorological changes[5] or by epidemic diseases;
c) Long-period cycles: when changes are not mutually independent and they produce their effect on a long term and a cycle period of several years may be assumed. They are usually correlated to general economic conditions.

In the present paper trends as well as long-term cycles are disregarded and only short-term fluctuations are taken into account. For this purpose a structure variable will be introduced to represent short-term fluctuations in the number of claims. In practice the (deterministic) parameter of the simple Poisson distribution for the number of claims of year t will turn to be a stochastic parameter $n_t \cdot \widetilde{q}$, where $\widetilde{q}$ is a random structure variable[6] having its own probability distribution depending on the short-term fluctuations it is going to represent. If no trends are assumed, the only restriction for the probability distribution of $\widetilde{q}$ is that its expected value has to be equal to 1.
The presence of this second source of randomness will clearly increase the standard deviation in the number of claims $\widetilde{k}_t$ and very often the skewness will be greater, thus increasing the chance of excessive claim numbers.

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4 See Beard, Pentikainen and Pesonen (1984).

5 Seasonal fluctuations are here disregarded because annual results are investigated. They should be clearly taken into account if results were analysed on a six-months basis (for instance when variation in claim frequencies between summer and winter are present in a motor insurance portfolio).

6 Here the variation of the Poisson parameter n from one time unit to the next is analysed. It is worth to recall that a "structure variable" have been also used when the variableness of the Poisson parameter from one risk unit to the next is to be investigated (see Bűhlmann (1970)).

In the following a Gamma distribution will be assumed as the probability distribution of the structure variable $\tilde{q}$. Then a negative binomial distribution is obtained for the random number of claims. Under these assumptions $\tilde{X}_t$ is denoted to be a compound Polya Process, as a special case of the more general compound Mixed Poisson Process. In this particular case, the moments of the structure variable q are given by:

$$E(\tilde{q}) = 1 \qquad \sigma(\tilde{q}) = 1/\sqrt{h} \qquad \gamma(\tilde{q}) = 2/\sqrt{h}$$

A usual estimate of h is the reciprocal value of the observed variance of q.

The claim amounts, denoted by $\tilde{Z}_{i,t}$, they are assumed to be i.i.d. random variables with a continuous distribution - having d.f. S(Z) – and to be scaled by only the inflation rate in each year. The moments about the origin are equal to:

$$E(\tilde{Z}_{i,t}^{j}) = (1+i)^{j \cdot t} \cdot E(\tilde{Z}_{i,0}^{j}) = (1+i)^{j \cdot t} \cdot a_{jZ,0}$$

with $\tilde{k}_t$ and $\tilde{Z}_{i,t}$ mutually independent for each year t.

The expected claim size has been simply denoted by m whereas r_{2Z} and r_{3Z} are risk indices of the claim size distribution[7]. Further, the skewness of the aggregate claim amount is reducing (increasing) time by time accordingly the positive (negative) real growth rate g as a natural result of the Central Limit Theorem.

The risk reserve ratio $\tilde{u}_t = \tilde{U}_t / B_t$ is usually preferred to be analysed instead of the risk reserve amount, and its equation (disregarding for the moment the reinsurance effect) is given by:

$$\tilde{u}_t = r \cdot \tilde{u}_{t-1} + p \cdot \left[(1+\lambda) - \frac{\tilde{X}_t}{P_t} \right] \tag{3}$$

where r and p denote the following two non negative joint factors:

$$r = \frac{1+j}{(1+i)\cdot(1+g)} \qquad\qquad p = \frac{1-c}{1+\lambda}(1+j)^{1/2} = \frac{P}{B}\cdot(1+j)^{1/2}$$

The annual factor r is depending on the investment return rate j, the claim inflation i and the real growth rate g; on the other hand factor p is depending on the incidence of the risk premium by gross premium (P/B), constant if expenses

7 Risk indices of the claim size distribution are:

$$r_{2Z} = \frac{a_{2Z}}{(a_{1Z})^2} = \frac{a_{2Z}}{m^2} \qquad \text{and} \qquad r_{3Z} = \frac{a_{3Z}}{(a_{1Z})^3} = \frac{a_{3Z}}{m^3}.$$

and safety loading coefficients (c and λ) are maintained constant along the time, increased of the investment return for half a year.
After some manipulations, the stochastic equation (3) of the ratio $\tilde{u}_t$ turns to:

(4) $$\tilde{u}_t = r^t \cdot u_0 + p \cdot \left[(1+\lambda) \cdot \sum_{h=0}^{t-1} r^h - \sum_{h=1}^{t} \frac{\tilde{X}_h}{P_h} \cdot r^{t-h} \right]$$

The expected value of this capital ratio can be easily derived:

(5) $$E(\tilde{u}_t) = \begin{cases} u_0 + \lambda p \cdot t & if \quad r = 1 \\ r^t \cdot u_0 + \lambda p \cdot \dfrac{1-r^t}{1-r} & if \quad r \neq 1 \end{cases}$$

As well known in actuarial literature[8], if r=1 the expected value of the ratio $\tilde{u}_t$ is a straight line by time t (linear increase if loading coefficient λ is positive) whereas if r≠1 a non linear behaviour of the expected value of risk reserve ratio is realised. It is worth to emphasize $E(\tilde{u}_t)$ initially depends significantly on the initial ratio u_0 (by the factor r^t) but in case r<1 its dominance is shortly decreasing in favour of the second addendum, where the safety loading coefficient λ is playing a key role. Besides, the dimension n of the Insurer is not affecting the expected level of the capital ratio U/B; actually n is affecting both risk reserve U and premiums volume B but according to the same proportion (at least in the present model) and then the expected ratio is independent from the number of policies in force.
Further, only if r<1 is fulfilled a finite convergence level of the expected ratio is obtained ("equilibrium level"):

(6) $$\bar{u} = \lim_{t\to\infty} E(\tilde{u}_t) = \frac{\lambda \cdot p}{1-r}$$

Some comments are needed here. Firstly, not only the dimension n but also the initial value u_0 do not contribute to the equilibrium level $\bar{u}$. Secondly, whether that equilibrium level is either improving or not over the initial value u_0 is depending on the input parameters λ, c and r, whereas higher safety loading and investment return clearly will drive up the equilibrium level while higher expenses, real growth and claim inflation will depress it.

[8] At this regard see Pentikäinen and Rantala (1982) and Beard et al. (1984).

In case r≥1 the expected ratio $E(\tilde{u}_t)$ diverges to positive or negative infinite values according to sign and value of the safety loading coefficient.
As regards variance, skewness and higher moments of the risk reserve ratio described in equation (4) see Pentikainen & Rantala (1982) and, for further results, Savelli (2002a).

3. A MEASURE OF RISK IN INSURANCE SOLVENCY ANALYSES.

In practical analyses when different management strategies may be pursued in the market then it is necessary to compare multiple pairs of risk and return measures. A traditional approach is to face the problem according to a mean-variance efficient frontier, where the best strategy for the time horizon is to maximize the expected value of the risk reserve (or either the ratio U/B) once fixed the initial capital (equivalent to maximize the return for stockholders) and, at the same time, to minimize its variance.
The main shortcoming in using the variance as a risk measure is that according to this view the risk is entailed in all deviations from the mean, without any reference to the algebraic sign. Very often in insurance and finance the real risk is only the downside risk and then a semi-variance approach would be preferred in spite of "favourable" signs are not counted for in the risk measure.

Another well known one-sided approach to risk evaluation is the Value-at-Risk (VaR) widely used when the risk relies on the occurrence of unfavourable events such as insolvency are to be estimated. That kind of approach has a sound background in actuarial literature, where Capital-at-Risk (CaR) and probability of ruin have usually been the main pillars in solvency analyses.
In insurance solvency, CaR can be summarized as the expected maximum loss for an insurer over a target horizon within a given confidence level (e.g. 99%); in other words it denotes a monetary amount for the risk of managing an insurance company.
The CaR for the horizon time (0,t) given a 1-ε confidence level is then given by:

$$CaR(0,t) = U_0 - U_\varepsilon(t)$$

where $U_\varepsilon(t)$ is the ε-th quantile of the Risk Reserve amount at time t (with a confidence level required in the analysis rather large, at least 95%) and U_0 the initial risk reserve.

In the present framework both premium volume and investment return are deterministic and $U_\varepsilon(t) = u_\varepsilon(t) \cdot B(t)$ is always fulfilled. If a CaR measure compared to initial capital U_0 is preferred, the following ratio can be easily obtained:

$$u_{CaR}(0,t) = \frac{CaR(0,t)}{U_0} = 1 - u_\varepsilon(t) \cdot \frac{B_t}{U_0} = 1 - \frac{u_\varepsilon(t)}{u_0} \cdot \frac{(1+j)^t}{r^t}$$

as a relative measure of the initial risk capital under the risk of default. In practice, it gives the measure of the CaR as a percentage of the initial capital when it is a non-zero amount. It is obviously dependent on the time horizon (0,t) as well as the requested confidence level 1-ε, increasing as much as confidence level and time horizon are greater.

Another analogous measure for insurance solvency analyses is the minimum risk-based capital (possibly required by either regulators and/or legislation) in order to be still in a solvency state after t years since the valuation date ($U_t \geq 0$) within a given 1-ε confidence level. In practice, disregarding the presence of an initial risk capital, in the present framework this measure (here denoted by U_{Req}) should fulfil the following equation:

(7) $$(1+j)^t \cdot U_{Req}(0,t) = -U_\varepsilon(t)$$

where U_ε is always the ε-th order quantile of the risk reserve distribution in year t. The amount $U_{Req}(0,t)$ is then the minimum capital to be required as a buffer to ensure that maximum losses accumulated by the insurer until year t (within a 1-ε confidence level) can be offset and a solvency state still maintained. For supervisory purposes the time horizon t is mostly included between 1 and 2 years.

In simulation models an initial risk capital (or a capital ratio) is usually given as an input parameter of the model, and in order to use the simulation results to draw up the measure $U_{Req}(0,t)$, reminding that investment return is here assumed deterministic and then the total financial contribution of the initial capital is deterministic as well, the equation (7) can be easily used in the equivalent form:

(7bis) $$(1+j)^t \cdot U_{Req}(0,t) = -\left[U_\varepsilon(t) - U_0 \cdot (1+j)^t\right]$$

In practice this means that initial required capital, plus compound interests, are sufficient to cover (within a 1-ε confidence level) the unfavourable deviations

(losses) among the input initial capital (increased by compound interests) and the quantile of the risk reserve distribution at year t.
Therefore, through formula (7bis) the minimum capital required at time 0 in order to be assured the state of solvency at year t (always with a confidence level of 1-ε) is obtained:

(8) $$U_{\mathrm{Re}q}(0,t) = -\left[(1+j)^{-t} \cdot U_{\varepsilon}(t) - U_0\right]$$

Expressed as a ratio of the initial gross premiums volume (B_0), that risk measure can be written as[9]:

(9) $$u_{\mathrm{Re}q}(0,t) = \frac{U_{\mathrm{Re}q}(0,t)}{B_0} = u_0 - \frac{u_{\varepsilon}(t)}{r^t}$$

A possible shortfall in these two approaches is that a simple quantile of the U/B probability distribution at year t (partly) disregards the state of the simulation paths in the previous years (1, 2,…t-1). For instance, it occurs when the simulation paths of ratio U/B are partly negative at time t-1 and afterwards all of them are positive at the next time point t, from which a favourable quantile U_{ε} (positive and rather large) is drawn up for year t; and consequently no minimum capital might be required if the time horizon is fixed at year t, notwithstanding the probability to be in state of ruin at year t-1 is not zero.

To avoid such kind of drawbacks, it could be useful to analyse simultaneously these quantile measures and the probability of ruin for the time horizon (0,t) which takes into account also ruins occurred at years t-1, t-2 …1.
Given the initial capital U_0=U and defined the state of ruin when the Risk Reserve is negative, let denote by $\varphi(U;t)$ the "probability to be in ruin state at year t" irrespective of the (ruin or not-ruin) state at previous years (t-1, t-2,…1):

(10) $$\varphi(U;t) = \Pr\left(\tilde{U}_t < 0 \ / \ U_0 = U\right)$$

On the other hand, the "finite time ruin probability" in the time span (0,T) is the probability to be in ruin state at least in one of the time points 1, 2 … T-1 and T[10]:

[9] According the mentioned framework a simple relation among u_{CaR} and u_{Req} is in force:

$$u_{\mathrm{Re}q}(0,t) = u_0 - (1+j)^{-t}\left[1 - u_0 \cdot u_{CaR}(0,t)\right] = u_0 - (1+j)^{-t}\left[1 - \frac{CaR(0,t)}{B_0}\right]$$

[10] Natural lower and upper limits for the time span ruin can be derived:

$$\max_{t \in (0,T)} \varphi(U;t) \le \psi(U;T) \le \sum_{t=1}^{T} \varphi(U;t)$$

(11). $$\psi(U;T) = \Pr\left(\widetilde{U}_t < 0 \quad for \quad at \quad least \quad one \quad t = 1,2,...T \quad / \quad U_0 = U \right)$$

and consequently the survival probability $\Phi(U,T)$ is given by:

$$\Phi(U;T) = 1 - \psi(U;T) = \Pr\left(\widetilde{U}_t \geq 0 \quad for \quad each \quad t = 1,2,...T \ / \ U_0 = U \right)$$

Finally, the "one-year ruin probability" $\psi(U;t-1,t)$ is the probability to fail in a ruin state for the first time at the time point t, having been in a no-ruin state for all the previous years:

$$\psi(U;t-1,t) = \Pr\left\{ \widetilde{U}_t < 0 \ \ and \ \ U_h \geq 0 \quad for \quad h = 1,2,...,t-1 \right\}$$

It can be also derived from the finite-time ruin probabilities by the well known relation:

$$1 - \psi(U;t) = [1 - \psi(U;t-1)] \cdot [1 - \psi(U;t-1,t)]$$

from which the one-year ruin probability is easily obtained:

(12) $$\psi(U;t-1,t) = 1 - \frac{1 - \psi(U;t)}{1 - \psi(U;t-1)}$$

In case the ruin barrier is otherwise defined than $U_{RUIN}(t)=0$, the above mentioned ruin probabilities will be consequently modified. For instance, an alternative ruin barrier can be assumed to be either the EU Minimum Solvency Margin of the year (MSM_t) or the Guarantee Amount equal to one third of the EU MSM of the year:

$U_{RUIN}(t) = MSM(t)$ or $U_{RUIN}(t) = 1/3*MSM(t)$.

As well as the CaR or VaR approaches (based on quantile values) the main shortcoming of the ruin probability as a risk measure is that it does not say anything on the left tail of risk reserve probability distribution and which is the most likely amount of ruin shortfall $U_{RUIN}(t) - U(t)$ to occur. In practice, as well as quantile measures, also for ruin probability no measure is given on the probability distribution of the shortfall (the amount of ruin).

More recently, Artzner et al. (1999) have pointed out how in the use of quantile measures the size of the loss is not taken properly into account, because no reference is made at the shape of the tail distribution exceeding the quantile. Indeed, in two different cases having the same VaR (or equivalently CaR or ruin probability) it may happen to get different expected shortfalls. Moreover, the same authors advocated calculating the Conditional Tail Expectation (CTE) as a more useful risk measure taking into account either the probability of the event occurring and the magnitude of the resulting loss.

In practice, a similar approach is here preferred (see Haberman et al. (2003)) in order to device a suitable measure of solvency risk: the Unconditional Expected Shortfall.

Having denoted the event that a "ruin" occurs at year t for the insurer by $U_t<U_{RUIN}(t)$ (equivalent to $u_t<U_{RUIN}(t)/B_t=u_{RUIN}(t)$)[11] and the conditional Mean Excess Shortfall (the expected value of the ruin deficit) by MES(t), the Unconditional Expected Shortfall (UES) at year t is given by:

(13)

$$
\begin{aligned}
UES(U,U_{RUIN};t) = E\left[\max(0,U_{RUIN}-\tilde{U}_t)\right] = \\
&= \Pr\left(\tilde{U}_t < U_{RUIN}(t)/U_0 = U\right)\cdot E\left(U_{RUIN}(t)-\tilde{U}_t \;\; / \;\; \tilde{U}_t < U_{RUIN}(t)\right) = \\
&= \Pr\left(\tilde{u}_t < u_{RUIN}(t)\right)\cdot \left[E\left(u_{RUIN}(t)-\tilde{u}_t \;\; / \;\; \tilde{u}_t < u_{RUIN}(t)\right)\cdot B_t\right] = \\
&= \varphi(U,U_{RUIN};t)\cdot MES(t)
\end{aligned}
$$

In the recent actuarial literature is emphasized how this risk measure can be regarded as the risk premium of an insurance contract which would cover the shortfall of the company in case it occurs, being the sum of the expected shortfalls (ruin deficits) weighted by their probabilities.

Furthermore, the UES is a one-sided risk measure, like the semi-variance, in which deficits are included but surpluses are ignored.

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[11] As mentioned before a state of ruin can be established also if the risk reserve, notwithstanding positive, is under the level of a positive amount, as the EU minimum solvency margin (MSM) or other correlated measures as the Guarantee Amount (=1/3*MSM). For management purposes the ruin barrier may be fixed also at a certain percentage of premium volume (e.g. 5% of gross premiums) as an early warning level. That may be preferred when medium or long-term analyses are carried out. As regards some other definitions of a "failure" state for an insurer see Coutts and Thomas (1997).

4. A MEASURE OF PERFORMANCE AND SOME RELATIONS WITH THE CAPITAL LEVEL.

Here the expected Return on Equity (RoE) will be preferred as a measure for the Insurer's performance and let $\overline{R}(0,T)$ denote the expected RoE all over the full time horizon (0,T). In the framework here assumed, where no dividends are present, it will be equal to:

$$(14) \qquad \overline{R}(0,T) = E\left(\frac{\widetilde{U}_T - U_0}{U_0}\right) = (1+g)^T (1+i)^T \cdot \frac{E(\widetilde{u}_T)}{u_0} - 1$$

In case the joint factor r is less than 1, reminding the formula of the expected capital ratio at time T and its asyntotic value (i.e. the "equilibrium level"):

$$E(\widetilde{u}_T) = r^T \cdot u_0 + \lambda p \frac{1-r^T}{1-r} \qquad \overline{u} = \lim_{T\to\infty} E(\widetilde{u}_T) = \frac{\lambda p}{1-r}$$

another relevant expression that may be used is:

$$\overline{R}(0,T) = (1+g)^T (1+i)^T \cdot \left[r^T + (1-r^T)\cdot\frac{\overline{u}}{u_0}\right] - 1$$

Once the finite-time expected RoE is computed we can also derive the expected forward rate $\overline{R}fw(t-1,t)$ from the well-known recursive equation:

$$\left[1+\overline{R}(0,t-1)\right]\cdot\left[1+\overline{R}fw(t-1,t)\right] = 1+\overline{R}(0,t)$$

from which (being no dividends assumed), the following rate (gross of taxation) is obtained[12] :

$$\overline{R}fw(t-1,t) = \frac{1+\overline{R}(0,t)}{1+\overline{R}(0,t-1)} - 1 = (1+g)(1+i)\cdot\frac{E(\widetilde{u}_t)}{E(\widetilde{u}_{t-1})} - 1 = j + \lambda p \cdot \frac{(1+g)(1+i)}{E(\widetilde{u}_{t-1})}$$

In our basic scenario it can be roughly approximated by the rate of investment return (j) plus approximately 2-3 times the safety loading coefficient (λ), where the value of the factor 2-3 is heavily affected by the reciprocal value of the expected initial capital ratio $1/E(u_{t-1})$. Further, a larger risk reserve will indeed depress the global relative return if the volume of business is not as larger and the

[12] Note that it does not necessary imply that:

$$1+\overline{R}fw(t-1,t) = E\left[\frac{1+\widetilde{R}(0,t)}{1+\widetilde{R}(0,t-1)}\right]$$

rate of profitability (λp) is unchanged. Moreover, the expected forward return will be monotonically decreasing if, as usual, the expected value of the ratio U/B is constantly rising up (here dividends to shareholders or fresh new capital are disregarded).

Another useful way is to present $\overline{R}fw(t-1,t)$ as function of either the initial capital ratio and the equilibrium level, as in the next formula:

$$(15) \qquad \overline{R}fw(t-1,t) = j + \lambda p \cdot \frac{(1+g)(1+i)}{E(\tilde{u}_{t-1})} = j + \frac{(1+g)(1+i)-(1+j)}{1+r^{t-1}\cdot(\frac{u_0}{\overline{u}}-1)}$$

It shows clearly as the expected forward RoE is only depending on the next six market and insurer no-dimensional parameters:

- initial capital (by u_0);
- safety loading (by λ);
- expenses loading (by the presence of c in the parameter p);
- real growth (by g);
- claim inflation (by i);
- investment return (by j).

Furthermore, a strong relationship between return and capital for a general insurer is included. If r<1 the expected forward annual return has a limit for the time going to infinity only depending on the real growth and the claim inflation:

$$\overline{R} = \lim_{t\to\infty} \overline{R}fw(t-1,t) = (1+g)(1+i)-1$$

In case the monotony of expected value of the capital ratio U/B[13] is fulfilled, as it is in our framework where no shock effects or business cycles are regarded, then the monotony of the expected annual return is assured too (as confirmed by the formula (15) where the only time-dependent factor is the joint factor r – assumed to be minor than 1).
In practice, if the starting value of the capital ratio is higher (lower) than the equilibrium level then the expected annual forward RoE will be monotonically increasing (decreasing) until its limit value:

if $u_0 > \overline{u}$ then $\overline{R}fw(t-1,t)$ monotonically increasing to $(1+g)(1+i)-1$
if $u_0 < \overline{u}$ " $\overline{R}fw(t-1,t)$ " decreasing to $(1+g)(1+i)-1$

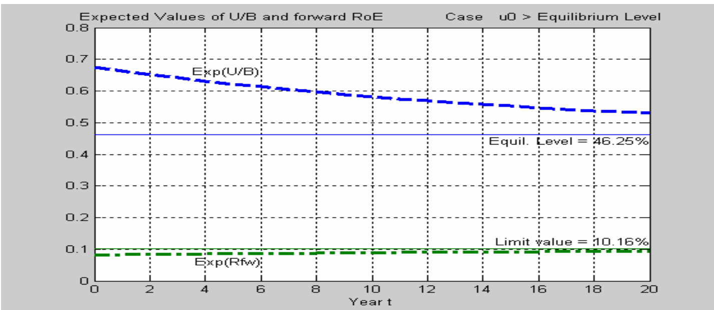
13 Monotonically either increasing or decreasing to the finite equilibrium level $\overline{u} = \lim_{t\to\infty} E(\tilde{u}_t) = \frac{\lambda \cdot p}{1-r}$ if r<1. Consequently, if the initial capital ratio U/B will be lower (higher) than the mentioned equilibrium level, the expected value of U/B will be monotonically increasing (decreasing) towards that value.

This relation displays the strong relation between capital and return and the key role of the capital ratio available at the valuation time (u_0):

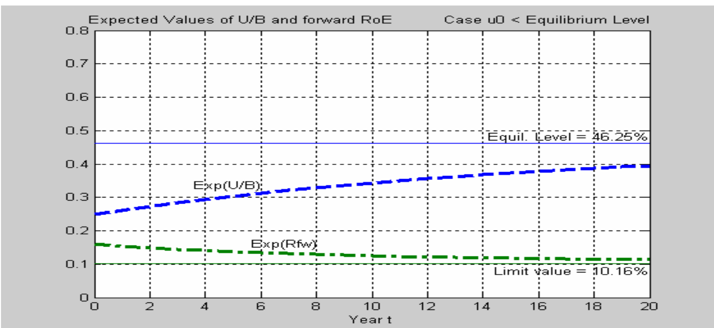
- if $u_0 > \bar{u}$ (see **Figure 1**, upper graph) the insurer would expect an increasing annual forward RoE for the next years until the upper limit $\bar{R}$. Meanwhile, a depressed capital ratio u_t will be expected along the time, towards the bottom limit $\bar{u}$ (represented by the equilibrium level), and consequently a larger risk of ruin will be likely in force year by year reminding also the rising variability of the time process;
- vice-versa (see Figure 1, lower graph), if the insurer has available only an initial capital ratio $u_0 < \bar{u}$, the annual forward RoE will be expected to be decreasing (but clearly at a higher level than in the previous case $u_0 > \bar{u}$) in the time horizon until the lower limit $\bar{R}$ (that is the upper limit in the previous case). On the other hand, the capital ratio u_t will be expected to increase until the upper limit $\bar{u}$ and consequently it might reduce its risk of ruin year by year in case this increase were sufficient to offset the rising variability of the process over the time.

FIGURE 1: Comparison between the Expected Value of capital ratio U/B and the Expected value of the forward rate RoE (time horizon of 20 years)
Parameters: λ=3.5% c=25% g=8% i=2% j=4%

Case $u_0 > \bar{u}$: u_0=67.5%



Case $u_0 < \bar{u}$: u_0=25.0%



In Figure 1 an example is figured out in both cases for these expected values in a rather general scenario with a particular set of parameters. These parameters identify a limit value for the expected capital ratio U/B equal to 46.25% ("equilibrium level") and a limit value of the expected forward rate of RoE equal to 10.16%.
For instance, if the initial capital ratio u_0 is 67.5% then E(U/B) is decreasing to 62.19% at year 5 and to 52.98% at year 20 whereas E(Rfw) is increasing from the initial rate of 8.14% to 8.51% at year 5 and to 9.34% at year 20.
On the other hand, if u_0 is 25% then E(U/B) is increasing to 30.31% at year 5 and to 39.53% at year 20 whereas E(Rfw) is decreasing from the initial rate of 16.00% to 13.70% at year 5 and to 11.28% at year 20.

5. THE PARAMETERS OF A THEORETICAL STANDARD INSURER.

In the present section some results of the simulation model illustrated in the previous sections are reported for a theoretical Standard Insurer in a time horizon of five years (T=5).
The features of this Standard Insurer are not drawn by a real data set but are assumed in order to be representative of a small insurer with a single line of business made by a rather homogenous personal line as Motor Third-Party Liability insurance (MTPL). The assumed parameters of this Standard Insurer are reported in the next **Table 1** and the results of 2.000 simulations of the ratio U/B are shown in **Figure 2** in order to give a first general idea on the simulation bundle and the risk profile of the insurer.

The parameters have been fixed in order to have as target a MTPL insurer with a standard deviation of its loss ratio X/B not far from 5% and a poor risk loading for competitive pressure. At this regard the safety loading coefficient λ has been fixed equal to 1.80% based on the standard deviation premium principle mentioned in Section 2, with coefficient β close to 0.28, applied at the portfolio in force at year 0.
In case the claim size variability were different from the present standard value c_Z=4, then the corresponding measure of the safety loading (if the same β is still regarded as suitable notwithstanding the different level of skewness and kurtosis), would have been approximately 1.5% if c_Z=2 and 3.1% in case of c_Z =10.

TABLE 1: Parameters of the Standard Insurer

Parameters :		**STANDARD INSURER**
Initial risk reserve ratio	u_0	0,250
Initial Expected number of claims	n_0	10.000
St.Dev. structure variable q	σ_q	0.05
Skewness structure variable q	γ_q	+ 0.10
Initial Expected claim amount (EUR)	m_0	3.500
Variability coeffic. of Z	c_Z	4
Safety loading coeffic.	λ	+ 1.80 %
Expense loadings coefficient	c	25.00 %
Real growth rate	g	5.00 %
Claim inflation rate	i	5.00 %
Investment return rate	j	4.00 %
Initial Risk Premium (mill EUR)	P	35,00
Initial Gross Premiums (mill EUR)	B	47,51
Joint factor (1+j)/(1+g)(1+i)	r	0,9433

It is worth pointing out that the coefficient λ is kept constant (1.80%) for the entire time horizon. Anyway, a periodical annual adjustment of the safety loading coefficient would have not affected significantly the results because of only a slight reduction of the ratio σ(X)/E(X) as the expected number of claims are increasing year by year (as an effect of the positive real growth of the premium volume), reminding that the pooling effect is less relevant when a not negligible structure variable (no diversifiable risk) is present as in this case (with σ_q =5%). In case a dynamic adjustment were adopted, the coefficient λ would have been slightly reduced time by time from 1.80% to 1.72% at year t=5.

In Figure 2 the upper dotted line represents the expected value of the EU Minumum Solvency Margin divided by the gross premiums amount whereas the lower dotted line is the classical Ruin Barrier U=0. As expected (see **Figure 2a)** the line MSM/B is rather constant over the time, close to 16.5% and rarely in excess of 17.5%, and its fluctuations are not so significant.
In case a Quota Share reinsurance treaty (with a ceding quota of 20%) were in force with a reinsurance commission at 20% of ceded premiums, the simulation values of the capital ratio U/B depicted in Figure 2 would turn to the values figured out in **Figure 3.**

FIGURE 2: results of 2.000 simulations of the ratio u=U/B for the **Standard Insurer** (u_0=25%, n_0=10.000, σ_q=5%, E(Z)=3.500, c_Z=4) – Results gross of reinsurance.

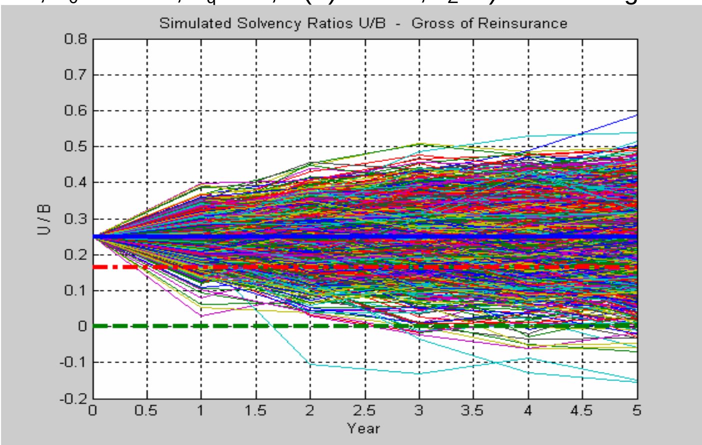


FIGURE 2a: results of 2.000 simulations of the ratio MSM/B for the **Standard Insurer** (u_0=25%, n_0=10.000, σ_q=5%, E(Z)=3.500, c_Z=4) – Results gross of reinsurance.

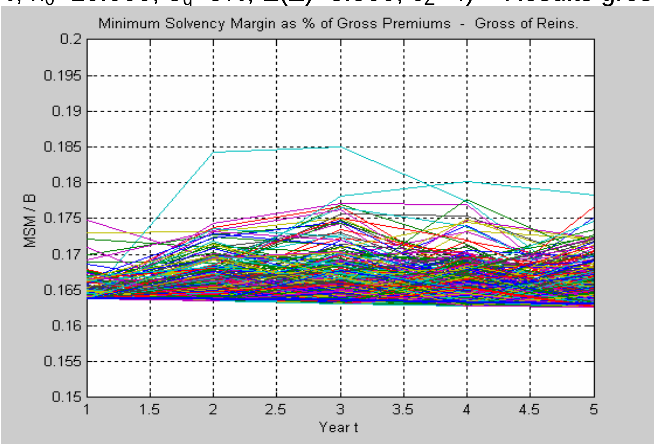
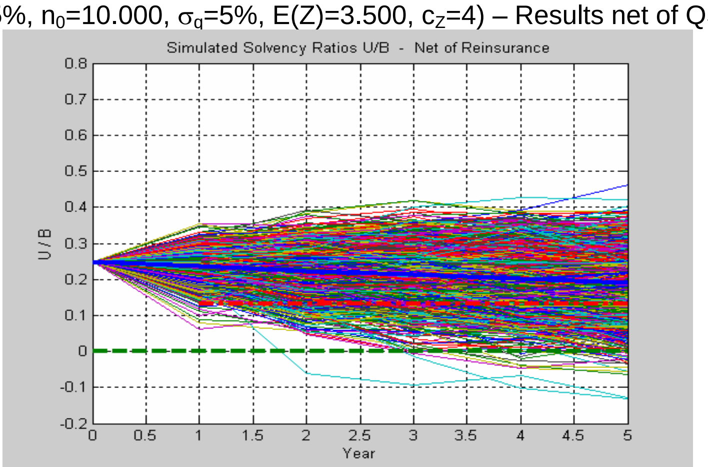


FIGURE 3: results of 2.000 simulations of the ratio u=U/B for the **Standard Insurer** (u_0=25%, n_0=10.000, σ_q=5%, E(Z)=3.500, c_Z=4) – Results net of QS reinsurance



Reins. Cover: Quota Share Reinsurance – Insurer retention 80% and reins. commission 20% of Ceded Premiums (c_{RE}=20%).

At a first glance the simulation bundle shows a reduced expected ratio of the capital ratio (for the unfavourable reinsurance commission) but on the other hand the variability of the bundle is clearly reduced and both ruin frequencies and ruin deficits are significantly lower.

6. THE RESULTS OF THE SIMULATION MODEL.

Clearly, the number of simulations needed for a more comprehensive risk analysis is by far larger than only 2.000; in the following, the results of 300.000 simulations are worked out for the assumed Standard Insurer.
In **Figures 4 and 5** the results of the simulations are reported with classical Monte Carlo scenarios. In Figure 4 the percentiles of the capital ratio U/B are figured out for some extreme percentiles (0.1%-1%-5%-50%-95%-99%-99.9%) gross and net of quota reinsurance and in Figure 5 the same results are presented with reference to lower and upper quartiles.

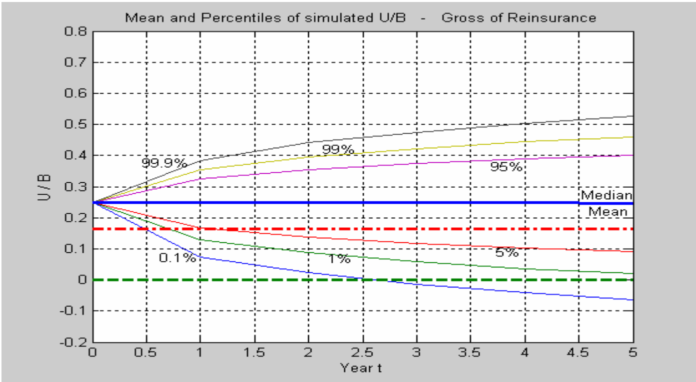
The main numerical results of the simulations are summed up in **Table 2**, where the main simulation moments of both the capital ratio U/B and the loss ratio X/B are reported. Instead, in **Table 3** the percentiles of those two indices are calculated.

TABLE 2: Standard Insurer – Results of 300.000 simulations
SIMULATION MOMENTS OF THE CAPITAL RATIO U/B AND PURE LOSS RATIO X/P

	SIMULATION MOMENTS OF U/B				SIMULATION MOMENTS OF X/P			
t	MEAN %	ST.DEV. %	SKEW.	KURT.	MEAN %	ST.DEV. %	SKEW.	KURT.
0	**25.00**				**100.000**			-
1	24.94	4.82	- 0.26	3.43	99.998	6.41	+ 0.25	3.43
2	24.88	6.60	- 0.18	3.25	99.993	6.37	+ 0.28	3.70
3	24.82	7.82	- 0.15	3.15	100.001	6.30	+ 0.26	3.68
4	24.78	8.73	- 0.13	3.12	99.983	6.24	+ 0.24	3.32
5	24.73	9.45	- 0.11	3.10	99.999	6.17	+ 0.23	3.31

Figure 4: Mean and Percentiles of U/B - Results of 300.000 simulations. Standard Insurer: u_0=25%, n_0=10.000, σ_q=5%, E(Z)=3.500, c_Z=4.
Percentiles figured out: 0.1% 1% 5% 50% 95% 99% 99.9%.

GROSS OF REINSURANCE



NET OF REINSURANCE (Quota Share with Ins. Retention 80% and c_{RE}=20%)

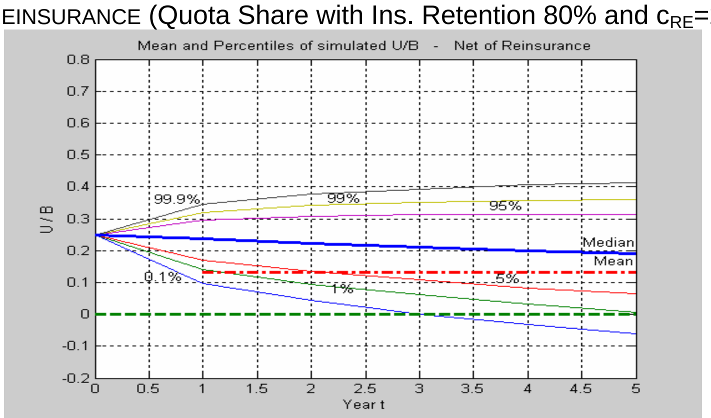


Figure 5: Lower and Upper Quartiles of the ratio U/B (Gross and Net of Reins.)
Standard Insurer: u_0=25%, n_0=10.000, σ_q=5%, E(Z)=3.500, c_Z=4.
Results of 300.000 simulations.

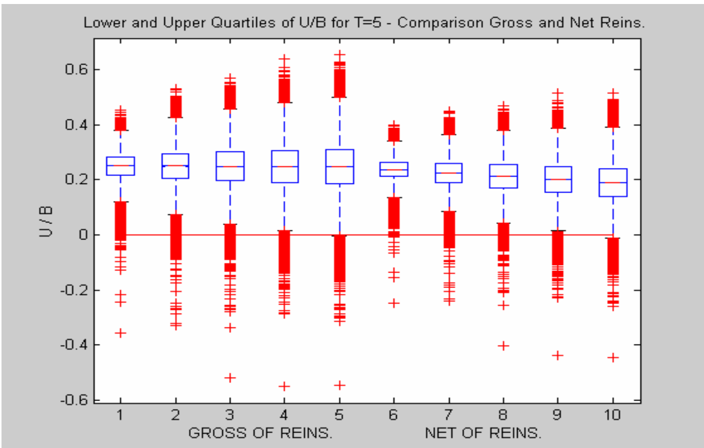


TABLE 3: Standard Insurer – Results of 300.000 simulations

SIMULATION MEAN AND PERCENTILES OF CAPITAL RATIO U/B AND PURE LOSS RATIO X/P

(% values)

Time	SIMULATION PERCENTILES OF U/B						SIMULATION PERCENTILES OF X/P					
t	MEAN	0.1%	1%	5%	MEDIAN	99.9%	MEAN	0.1%	1%	5%	MEDIAN	99.9%
0	**25.00**						**100.000**					
1	24.94	7.48	12.97	16.77	25.11	38.47	99.998	81.98	86.05	89.92	99.77	123.23
2	24.88	2.40	8.75	13.77	25.04	44.17	99.993	82.19	86.07	89.94	99.77	122.83
3	24.82	1.45	5.92	11.75	24.97	47.51	100.001	82.23	86.24	90.04	99.78	122.30
4	24.78	-4.17	3.62	10.19	24.95	50.44	99.983	82.44	86.36	90.13	99.78	122.03
5	24.73	-6.44	1.94	8.98	24.89	52.69	99.999	82.43	86.51	90.21	99.81	121.67

Moreover, in **Table 4** different ruin probabilities are reported for two different ruin barriers whereas in **Table 5** the associated expected return on equity is contained:

TABLE 4: Standard Insurer – Results of 300.000 simulations

RUIN PROBABILITIES

(% VALUES)

Time t	WITH RUIN BARRIER $U_{RUIN}=0$			WITH RUIN BARRIER U_{RUIN}=1/3*MSM		
	ANNUAL RUIN PROB.	ONE-YEAR RUIN PROB.	FINITE TIME RUIN PROB.	ANNUAL RUIN PROB.	ONE-YEAR RUIN PROB.	FINITE TIME RUIN PROB.
1	0.01	0.01	0.01	0.05	0.05	0.05
2	0.05	0.04	0.05	0.31	0.28	0.33
3	0.16	0.13	0.18	0.87	0.68	1.01
4	0.36	0.25	0.44	1.60	1.03	2.05
5	0.61	0.38	0.81	2.33	1.19	3.27

TABLE 5: Standard Insurer – Results of 300.000 simulations

EXPECTED RETURN ON EQUITY

(% VALUES)

Time t	FORWARD RATE	FINITE-TIME RATE
1	9.96	9.96
2	9.98	20.54
3	9.99	32.58
4	10.01	45.85
5	10.02	60.47

Because of the small variance and skewness of the process, it is to be emphasized these results of 300.000 simulations are extremely consistent with the exact value for the mentioned moments (at this regard see Savelli(2002b)). In case of an insurer with a much more variable portfolio (e.g. c_Z=10) and with a large skewness of the aggregate claim amount (e.g. +10) the required number of simulations to get the same consistency is obviously greater.

From the Figures 3 and 4 as well from numerical results reported in Tables 2, 3 and 4, the next main comments can be summed up:

- on a 1-year time horizon the simulated scenario for our Standard Insurer is rather promising as regards solvency, with a rather small probability at 0.01% (equivalent to a survival probability of 99.99%) to become ruined (U<0). Among other factors, it is clearly influenced by the initial capital ratio, assumed to be 25% (approximately 1.5 times the EU minimum solvency margin). Clearly worse or better results would have been obtained (as to the downside risk) if the initial capital were either lower (e.g. 20%) or higher (e.g. 30%);
- on an extended time horizon (0,T) the survival probability is clearly slightly decreasing as the required time horizon is larger: 99.95% for T=2, 99.82% for T=3, 99.56% for T=4 and finally 99.19% for T=5;
- the above mentioned survival probabilities are affected by the variability of the stochastic process defined by the risk reserve amount U (or U/B), in our framework depending by the variance of the aggregate amount of claims (X) or equivalently by the variance of the loss ratio (X/B). At this regard, it is worth to mention that for the assumption made at the present stage of the model the loss ratio X/B is equal to the pure loss ratio X/P multiplied for the factor P/B (depending only on safety and expenses loading, and then constant over the whole time horizon for the assumptions here made), factor always less than one. Then a crucial element is the variance of the random ratio X/P; for the Standard Insurer here investigated the standard deviation of this annual ratio is only slightly declining over the time because the increasing number of expected claims (by the factor 1+0.05) is not reducing significantly the variability by the time because of the presence of a non diversifiable factor (the structure variable has a significant standard deviation, assumed to be 5% in our simulations). From Table 2 the standard deviation of X/P is indeed only decreasing from 6.41% to 6.17% and it can be easily observed its impact on the variability of the ratio U/B at year 1, where a value of 4.82% is obtained, equal to 6.41% multiplied by roughly P/B (equal to 0.752);
- the shape of the U/B probability distribution is close to the Normal distribution, with special evidence at time 3-4-5. That is because of a skewness rather close to 0 (with values slightly decreasing from -0.3 to -0.1) and a kurtosis not so far from the value 3 of a normal distribution (decreasing from 3.4 to 3.1). Furthermore, reminding that in our framework the process U is defined by a sum of independent random variables (the annual aggregate

claims amounts X_t), it follows from the Central Limit Theorem that the ultimate limit for the probability distribution of U (and then of U/B) at infinity time is Normal distributed. The speed and the accuracy of the approximation are clearly depending on the parameters defining the risk reserve process. In case of a portfolio structure much more instable, i.e. with larger skewness and kurtosis (e.g. industrial risks), the approximation to the Normal distribution would be by far more unreliable and it might drive to undesirable underestimation of the risk;

- as regards the approximation of the U/B process to the Normal distribution, it is to be emphasized that usual short-cut formulas for minimum requirements or risk analysis making use of Normal-Power or other normal-related approximations must be carefully handled, bearing in mind that the reliability of the results is heavily affected by the higher order moments of the U distribution (skewness and kurtosis over all). It is beyond the goal of the present paper, but a comparison between the results of short-cut formulas and simulations would clearly put in evidence this crucial aspect in practical analysis;
- as regards profitability, for the moment we can observe that the expected value of the ratio U/B is deteriorating from 25% to approximately 24.73% (notwithstanding no dividends and no taxations are here regarded). The expected forward rate of RoE is almost constant around 10% with a negligible increase over the examined years from 9.96% of year 1 to 10.02% of year 5;
- finally, reinsurance covers could improve the risk profile of the Insurer, but are clearly depressing its profitability. For instance, if a 20% Quota Share reinsurance treaty would be in force all over the 5-years time horizon, with the Insurer receiving commissions equal to 20% of the ceded premiums (5% less than the general and acquisition costs annually afforded by the Insurer), the finite-time probability of ruin and the expected forward RoE would change significantly (see **Table 6**). In the depicted scenario the expected annual rate of return will be more than halved (see at year 1, where the expected RoE will turn from 9.96% to only 4.28%, just a little more than the investment return of the equity capital j=4%). Furthermore, the ruin probability will be improved with the mentioned cover but, as the value of the expected U/B net of reinsurance is faster decreasing on the bottom, for time t=5 the probability of ruin net of reinsurance (0.91%) will be higher than the figure obtained without any reinsurance (0.81%):

TABLE 6: FINITE-TIME EXPECTED ROE AND FINITE-TIME RUIN PROBABILITY
GROSS AND NET OF A QUOTA SHARE REINSURANCE (U_{RUIN}=0).
Standard Insurer – Results of 300.000 simulations

(% VALUES)

Time Horizon	GROSS. OF REINS.		NET OF 20% QS REINS.	
T	FINITE TIME EXP. ROE R(0,T)	FINITE-TIME RUIN PROB.	FINITE TIME EXP. ROE R(0,T)	FINITE-TIME RUIN PROB.
1	9.96	0.01	4.28	0.00
2	20.54	0.05	8.77	0.02
3	32.58	0.18	13.45	0.10
4	45.85	0.44	18.42	0.36
5	60.47	0.81	23.56	0.91

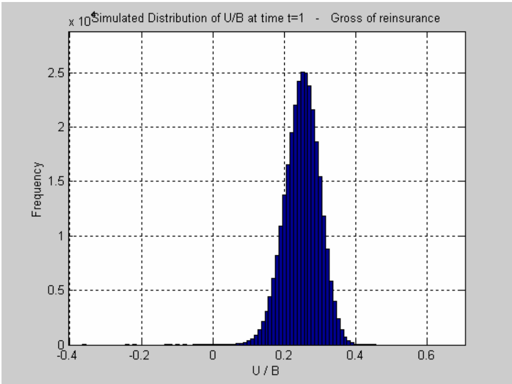
- the mentioned probabilities are referred to a ruin barrier U_{RUIN}=0. But in case one of the main risk target of the Insurer were to be compliant with the compulsory rules concerning the minimum solvency margin, in order to avoid as much as possible to claim fresh new capital from shareholders in the forthcoming years, the ruin barrier could be U_{RUIN}=MSM. In this case the probability to cross that new ruin barrier is obviously much higher; from Figure 4 it can be observed it results larger than 5% and increasing in the forthcoming years. Moreover, if a ruin barrier corresponding to the Guarantee Amount equal to 1/3 of the EU minimum solvency margin were in force, the results would be somewhere between the mentioned probability figures (see Table 4).

Moreover, in **Figure 6a** it is showed for our Standard Insurer the effect of the 20% Quota Share Reinsurance on the shape of simulation distributions of the capital ratio U/B at years 1-2-5.
The skewness values of the distributions gross and net of reinsurance are exactly the same for the proportional feature of the QS cover whereas the standard deviation of the ratio U/B net of reinsurance is exactly 80% of the standard deviation gross of reinsurance. Finally, as regards the expected values of the ratio U/B, the reinsurance cover is significantly reducing its value of approximately 5% at year 1, 10% at year 2 and 25% at year 5, because of the unfavourable reinsurance commissions.
On the other hand, in **Figure 6b** the simulation distributions of the capital ratio U/B at years 1-2-5 for the Standard Insurer are compared with the analogous distributions for a different Insurer (with a rough reference to a corporate liability line) having a more volatile portfolio (c_Z=10) with large skewness values. In both cases as the time increases the shape of the probability distributions becomes less skew but the variances are significantly increasing.

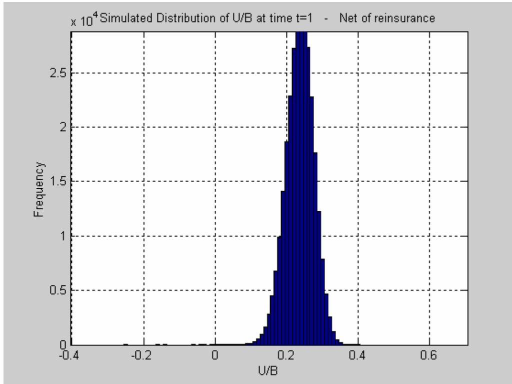
FIGURE 6a: Standard Insurer: comparison of the simulation distributions of the ratio U/B at years 1, 2 and 5 gross and net of a 20% Quota Share Reinsurance.
Results of 300.000 simulations.

Gross of Reinsurance

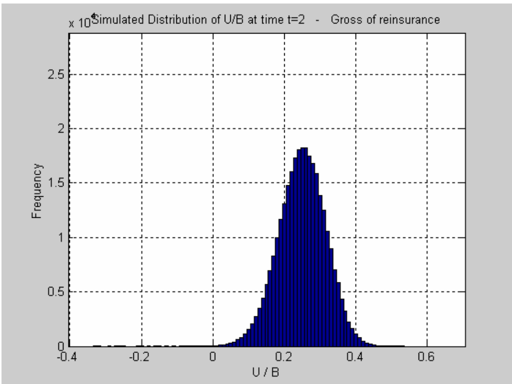
Net of QS Reinsurance
insurer retention quota = 80%
fixed reins. commissions = 20%



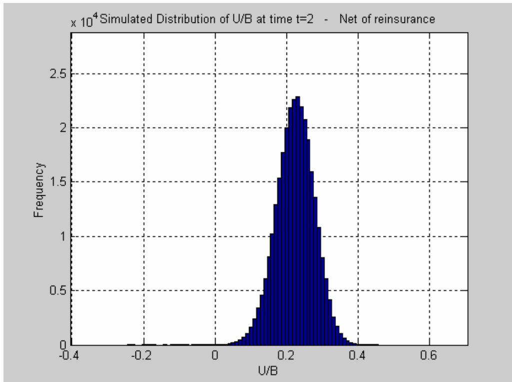
Year 1: mean=0.2494 std=0.0482 skew.=-0.26



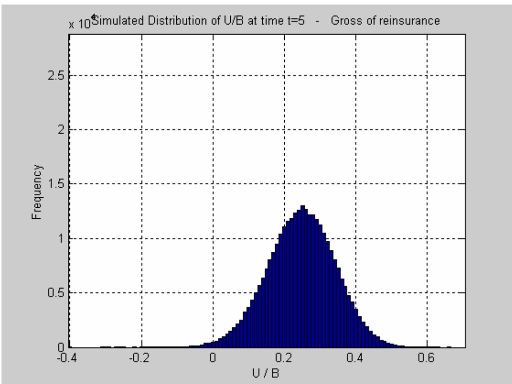
Year 1: mean=0.2365 std=0.0385 skew.=-0.26



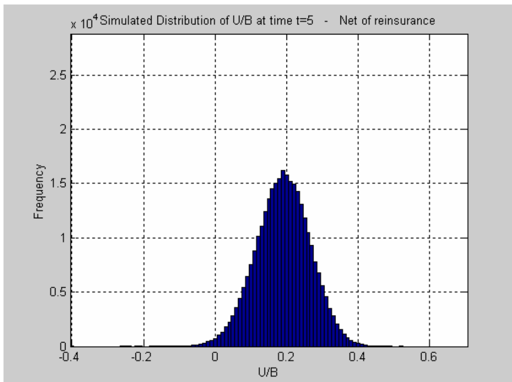
Year 2: mean=0.2488 std=0.0660 skew.=-0.18



Year 2: mean=0.2237 std=0.0528 skew.=-0.18



Year 5: mean=0.2473 std=0.0945 skew.=-0.11



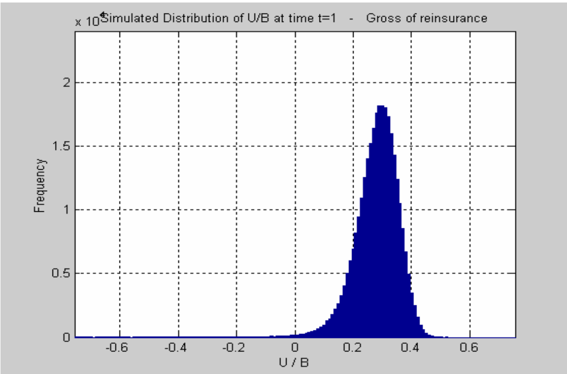
Year 5: mean=0.1896 std=0.0756 skew.=-0.11

Figure 6b: Comparison of the simulation distributions of the capital ratio U/B at years 1, 2 and 5 between Standard Insurer and a No Standard Insurer
Results of 300.000 simulations (Gross of Reins.)

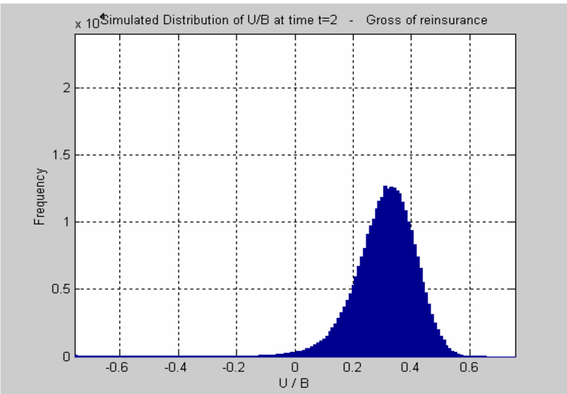
No Standard Insurer

u_0=25%, n_0=10.000, σ_q=5%,
E(Z)=**10.000**, c_Z=**10** and λ=**5%**

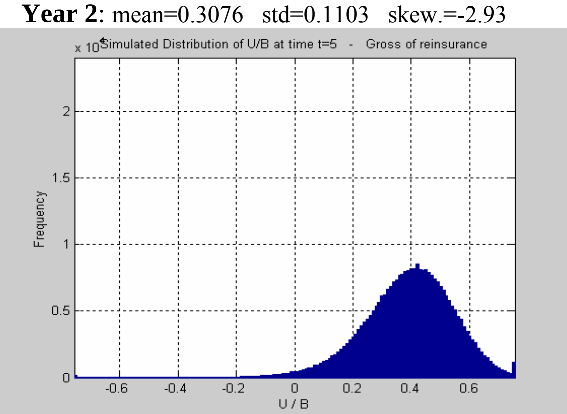
Standard Insurer



Year 1: mean=0.2792 std=0.0799 skew.=-4.79

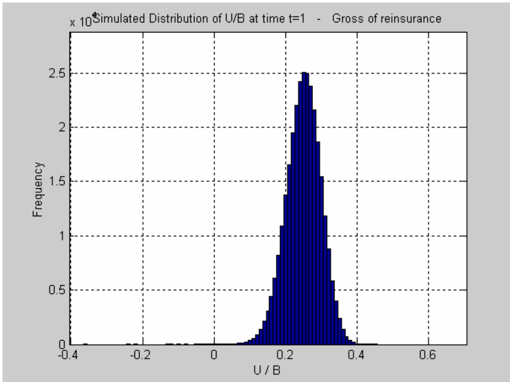


Year 2: mean=0.3076 std=0.1103 skew.=-2.93

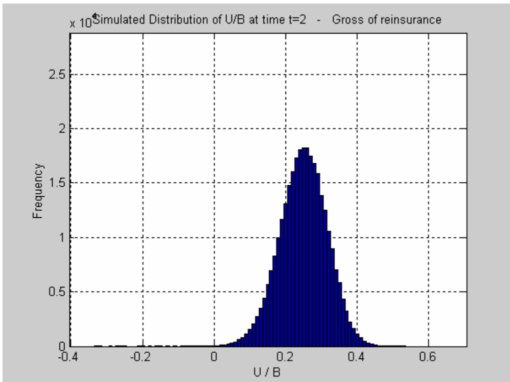


Year 5: mean=0.3878 std=0.1625 skew.=-1.66

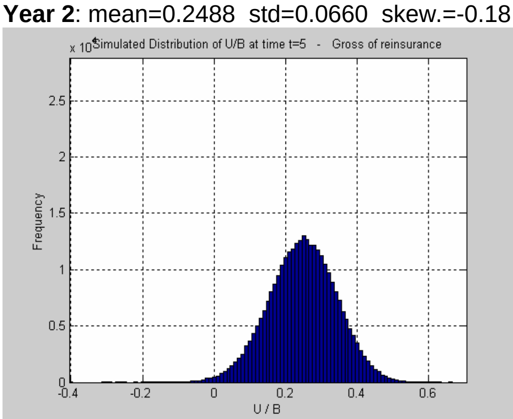
(MTPL line)
u_0=25%, n_0=10.000, σ_q=5%,
E(Z)=**3.500**, c_Z=**4** and λ=**1.8%**



Year 1: mean=0.2494 std=0.0482 skew.=-0.26



Year 2: mean=0.2488 std=0.0660 skew.=-0.18



Year 5: mean=0.2473 std=0.0945 skew.=-0.11

From the results of the simulation model we can also focus our attention to the amount of capital needed to be still in a solvency state at year t (within a given confidence interval). In Section 3 this amount has been denoted by $U_{Req}(0,t)$ and by the ratio $u_{Req}(0,t)$ if that required amount is divided by the initial gross premiums volume.

The results for our Standard Insurer (see **Table 7** and **Figure 7**) show how this measure is high sensitive to both confidence level and time horizon. Regarded only the cases T=1 and T=2 for supervisory targets, the required ratio with a confidence level of 99.0% and T=2 (15.2%) is not so far from the minimum ratio requested by the EU Minimum Solvency Margin (close to 16.4% in our case). As expected, the ratio $u_{Req}(0,T)$ is higher when a larger time horizon is preferred but according a non-convex behaviour. That means, as well known in actuarial literature that the short-cut approximation formula $u_{\mathrm{Re}q}(0,T) \cong \sqrt{T} \cdot u_{\mathrm{Re}e}(0,1)$ is roughly confirmed in the present assumed framework, where no autocorrelation between aggregate amount of claims has been assumed. Moreover, it is worth pointing out how for insurers with skewness and kurtosis higher than those here registered for the Standard Insurer this approximation is by far less reliable because of the leading factor is not anymore the standard deviation only.

Furthermore it is to be emphasized as for a more risky insurer, with a larger deviation of the loss ratio, would display results not so consistent with EU minimum capital requirements (e.g. when a more risky business line is regarded having a standard deviation of the loss ratio by far higher than 5%, e.g. industrial and corporate third-party liability).

TABLE 7: Standard Insurer – Results of 300.000 simulations
MINIMUM RISK CAPITAL REQUIRED FOR A DIFFERENT TIME HORIZON AS A PERCENT VALUE OF THE INITIAL GROSS PREMIUMS ($U_{REQ}(0,T)/B_0$) ACCORDING TWO DIFFERENT CONFIDENCE LEVELS

	GROSS OF REINS.			**NET OF REINS.**		
Time Horizon T	**CONFID. 99.9%**	**CONFID. 99.0%**	**$U_{REQ}(T)$ / $U_{REQ}(1)$**	**CONFID. 99.9%**	**CONFID. 99.0%**	**$U_{REQ}(T)$ / $U_{REQ}(1)$**
T = 1	17.07 %	11.26 %	1.00	14.73 %	10.09 %	1.00
T = 2	22.31 %	15.17 %	1.35	20.07 %	14.36 %	1.42
T = 3	26.73 %	17.94 %	1.59	24.83 %	17.80 %	1.76
T = 4	30.27 %	20.43 %	1.81	28.94 %	21.07 %	2.09
T = 5	33.62 %	22.40 %	1.99	32.99 %	24.01 %	2.38

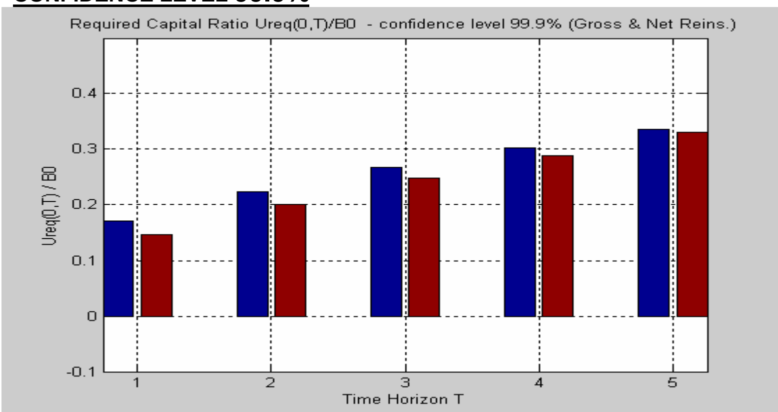
Finally, as regards the impact of the Quota Share Reinsurance, the simulation results show how the minimum required capital amount is greater than 80% of the amount required gross of reinsurance (as instead provided in the EU MSM formula). Concerning only the time horizon T=1 and T=2 the results show how this measure should be much more than 80% figure before mentioned, actually equal to roughly 85% for an annual horizon time and 90% for a two-years time span. It is worth to emphasize that it comes from reinsurance commissions of only 20% (against expenses for the Insurer equal to 25% of premiums), depressing the profit amount and consequently its solvency profile. Indeed, the shortcoming of the EU formula at this regard is to take into account only the actual retention of paid claims, but no reference is made to the treaty profitability (in a Quota Share the key role is played by the measure of the commissions), who heavily affects, as mentioned, the solvency level of the company also for short-time analyses, as shown in our case. Similar comments may be derived in case of scalar reinsurance commissions depending on the loss ratio.

FIGURE 7: Standard Insurer – Results of 300.000 simulations

MINIMUM RISK CAPITAL REQUIRED FOR DIFFERENT TIME HORIZON AS A PERCENT VALUE OF THE INITIAL GROSS PREMIUMS $u_{Req}(0,T)=U_{Req}(0,T)/B_0$

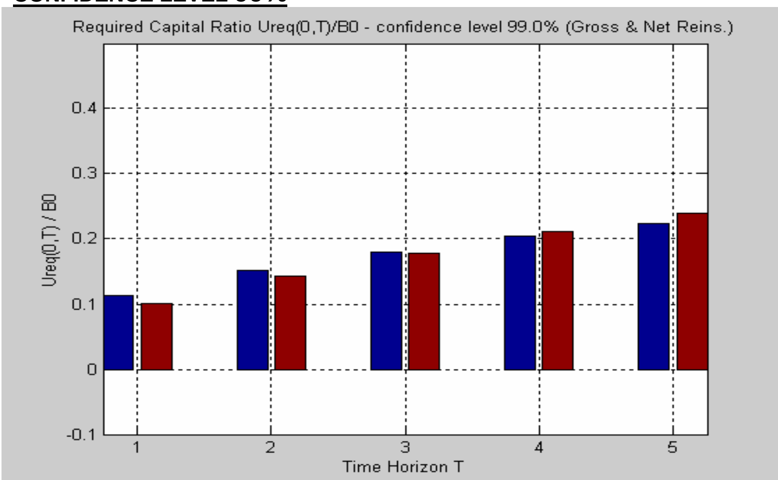
RESULTS GROSS AND NET OF Q.S. REINSURANCE

CONFIDENCE LEVEL 99.9%



Figures Gross of Reinsurance: 1st bar Figures Net of Reinsurance: 2nd bar

CONFIDENCE LEVEL 99%



Figures Gross of Reinsurance: 1st bar Figures Net of Reinsurance: 2nd bar

Even though to investigate on the fairness of the EU minimum solvency margin according to the features and the dimension of a general insurer is not the target of this paper, it is worth to mention that for the MTPL single-line small-dimensioned insurer here considered the EU solvency formula (approximately 16-18% of retained premiums) is sufficient to afford the unfavourable deviations in almost 99.9% but only regarding the pure underwriting risk. That suitability will be consistently reduced if either taxation and dividends will be regarded or also if other types of risks will be regarded, as e.g. financial and loss reserving risks, long-term cycles and catastrophic events.

7. RISK VS RETURN ANALYSES.

In **Figure 8** is illustrated the trade-off Risk vs Return of the examined Insurer for all the 5 years of the time horizon, where the measure of risk has been denoted by the Unconditional Expected Shortfall (UES) of the year (with two different ruin barriers) and as regards profitability by the finite time Expected RoE $\overline{R}(0,T)$.

A similar trade-off is represented also in **Figure 9** with two different ruin barriers but there the risk measure is the finite time ruin probability.

On these trade-off figures the following comments are to be pointed out:

- first of all both the risk measures are consistent each other. Actually, for the Standard Insurer here assumed and for the designed QS reinsurance treaty the slopes of the curves are very similar and they drive at the same strategies;
- as the time horizon is extended, the expected return $\overline{R}(0,T)$ is rising up as well as the risk measure (for either finite time ruin probability or unconditional expected shortfall) for both strategies (no reinsurance at all and the mentioned 20% QS reinsurance); comparing the results given by the model, in the designed framework the strategy to choose a 20% QS coverage is also efficient for any time horizon but not for T=5;
- actually, for T=1,2,3, or 4 in case the QS reinsurance strategy is preferred it would get less profitability and less risk of default (for both ruin barriers) but for the time horizon T=5 that is not the case: it would get a significant cut in the periodical profitability compared to no reinsurance strategy (23.56% instead of 61.12%) but notwithstanding that a larger measure of default risk is obtained (e.g. for the ruin barrier U_{RUIN}=0 the UES at year 5 is 0.237‰ of gross premiums, to be compared with the measure of 0.225‰ in case of no reinsurance). That is clearly due to the significant cut in profitability of the company as for a large portion of its business (20%) the technical profitability will be dramatically reduced for the unfavourable reinsurance commissions of

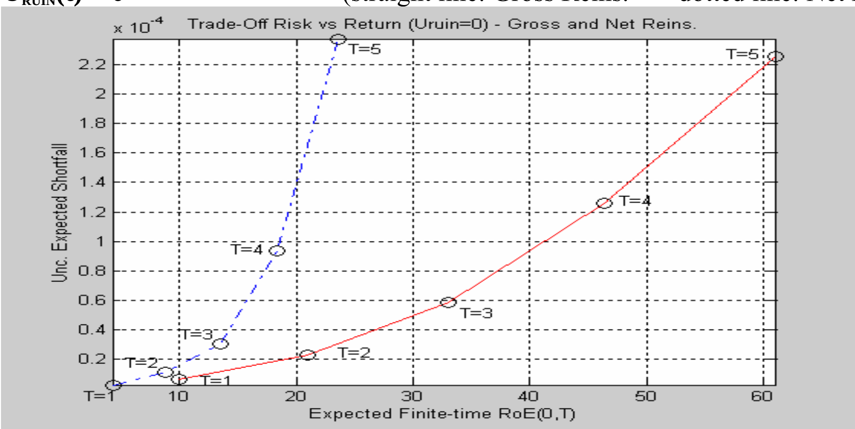
only 20% of the ceded premiums whereas the Insurer has a 25% level of expenses. On the long term this lack of profitability will bring the expected value of the capital ratio U/B closer to the ruin line and then a larger risk of insolvency will arise notwithstanding the process fluctuations are reduced by the reinsurance cover (see either Figure 4 and Table 6).

FIGURE 8: Risk vs Return trade-off for the Standard Insurer. Results of 300.000 simulations.

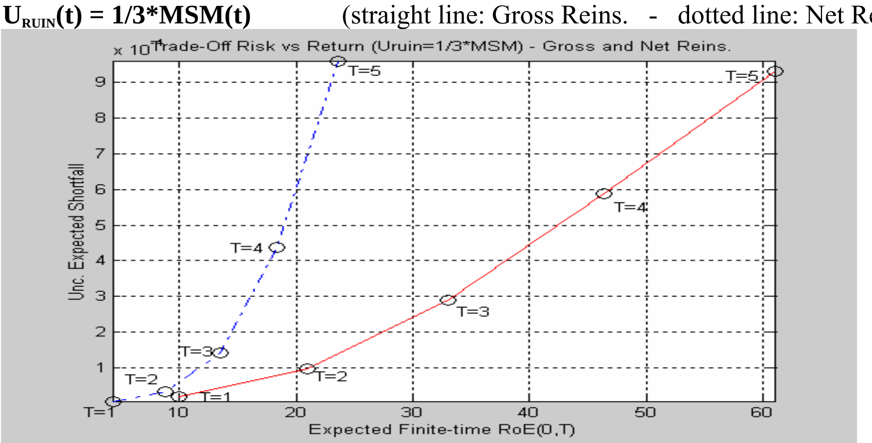
Risk measure: uncond. exp. shortfall / gross premiums $UES(T)/B_T$

Performance measure: finite time expected RoE $\overline{R}(0,T)$

$U_{RUIN}(t) = 0$ (straight line: Gross Reins. - dotted line: Net Reins.)



$U_{RUIN}(t) = 1/3*MSM(t)$ (straight line: Gross Reins. - dotted line: Net Reins.)



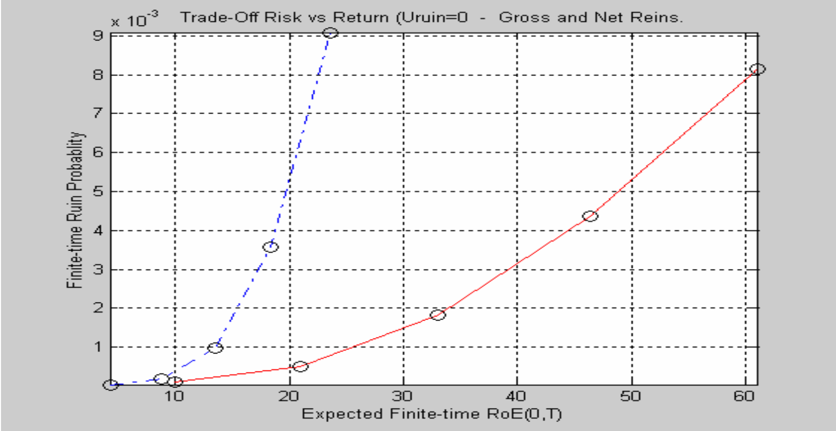
In the management practice usually the Insurer must choose, among different "efficient" strategies, according to a minimum target for its profitability and a maximum target for the risk measure (let say of UES with a classical ruin barrier U=0) and that can heavily affect the results. For instance, let assume that our Standard Insurer will decide his reinsurance strategy according to the estimated results at time 3, with the next min/max Insurer's constraints:

$\overline{R}_{MIN}(0;3)$ =25% and UES$_{max}$(3)=0.04‰*B$_T$ (the latter constraint equivalent, for instance, to a probability to be in ruin at year 3 equal to 1‰ combined with an expected value of the ruin deficit not exceeding 4% of gross premiums). In our case no practical strategy is available for the management fulfilling both targets: the no-reinsurance strategy has a periodical profitability by far larger than 25% but its UES is larger than 0.04‰ of premiums (0.058‰). On the other hand the 20% QS reinsurance strategy fulfils the requested upper limit of risk but its periodical profitability is rather poor (only 13.45% over the 3 years term).

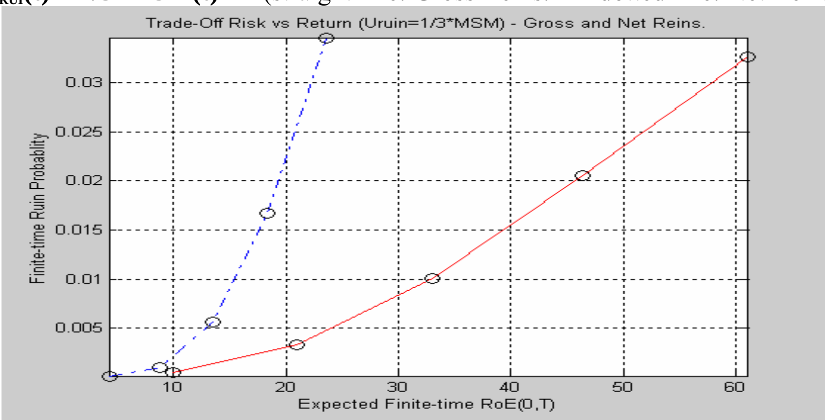
FIGURE 9: Risk vs Return trade-off for the Standard Insurer. Results of 300.000 simulations.

Risk measure:	finite time ruin probability	$\psi(0,T)$
Performance measure:	finite time expected RoE	$\overline{R}(0,T)$

U_{RUIN}(t) = 0 (straight line: Gross Reins. - dotted line: Net Reins.)



U_{RUI}(t) = 1/3*MSM(t) (straight line: Gross Reins. - dotted line: Net Reins.)



It is clear that the available 20% QS treaty (Treaty A) is not suitable for our Standard Insurer, first of all for its heavy lack of profitability and secondly for an unnecessary large cover.

Let assume now that on the reinsurance market there are two other reinsurance covers available for the company (with the identical risk of default estimated for the reinsurers involved):

- Treaty B (5% QS): a Quota Share reinsurance treaty providing for a small quota to be reinsured (5%) and a more favourable reinsurance commission (22.5%) compared with Treaty A;
- Treaty C (XL): is an Excess-of-Loss reinsurance treaty, providing the coverage (on a single claim basis) for each claim of the portfolio exceeding the single claim Insurer Retention M equal to EUR 115.000 (equal to the expected value of the claim size plus 8 times the standard deviation) and a safety loading equal to 10.8% of the ceded premiums (6 times the 1.8% figure applied by the Insurer).

It is worth pointing out that for both treaties B and C the amount of the ceded risk premium is exactly 5% of the total risk premiums P. Then both Reinsurers are expected to have a reinsured aggregate claim amount X_{RE} equal to 5% of the total aggregate claim amount X.

On the basis of these treaties, the new Risk-vs-Return simulation results are illustrated in **Figure 10**, where the ruin barrier is $U_{ruin}(t)=0$ and besides only the UES risk measure is figured out.

As expected, in case the Treaty B is taken into account the Risk/Return curves (no reinsurance and reinsurance) are closer each other compared to the Treaty A (20% QS): that is due to the fact that a smaller amount of risk is transferred (5% instead of 20%) and consequently either profitability and insolvency risk are closer to the values for no-reinsurance strategy. Moreover, the XL coverage (Treaty C) is more expensive than 5% QS coverage but it is more effective on reducing the downside risk (see also numerical values in **Table 8**).

Coming back to the management targets (R_{min}=25% and UES_{max}=0.04‰ for T=3), among the new two available treaties only the XL treaty strictly complies with the requested management constraints. Any way, it is worth to notice that also the 5% QS treaty is rather close to a full compliant of the constraints (in particular the risk measure - 0.0416‰ - is just a little bit over the 0.04‰ target measure and the expected return is significantly higher than the XL strategy, 30.41% vs 26.84%).

In spite of those mentioned reinsurance covers are theoretical, what explained here is a practical issue to assess the most appropriate reinsurance strategy according to a reliable valuation of the default risk and under the requested management constraints.

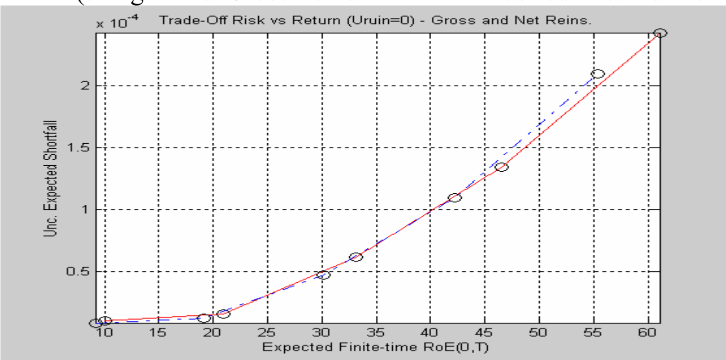
FIGURE 10: Risk vs Return trade-off for the Standard Insurer. Results of 300.000 simulations.

Risk measure: uncond. exp. shortfall / gross premiums $UES(T)/B_T$

Performance measure: finite time expected RoE $\overline{R}(0,T)$

Treaty B: 5% Quota Share with c_{RE}=22.5%

$U_{RUIN}(t)$ = 0 (straight line: Gross Reins. - dotted line: Net Reins.)



Treaty C: XL with M=EUR 115.000 and λ_{RE}=10.8%

$U_{RUIN}(t)$ = 0 (straight line: Gross Reins. - dotted line: Net Reins.)

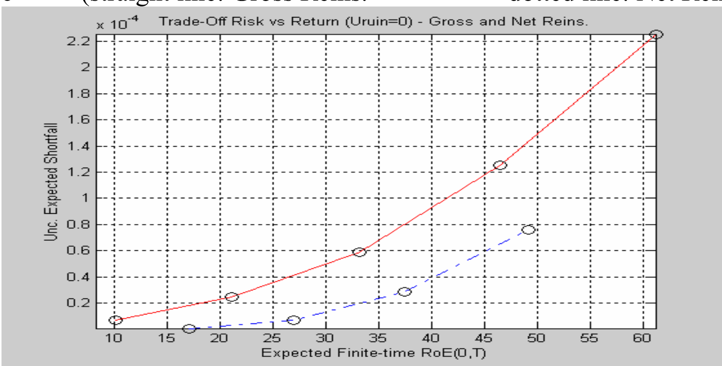


TABLE 8: Standard Insurer – Results of 300.000 simulations

RISK VS RETURN RESULTS FOR THE THREE THEORETICAL REINSURANCE TREATIES (% VALUES)

TIME HORIZON	NO REINSURANCE		TREATY A: 20% QS AND C_{RE}=20%		TREATY B: 5% QS AND C_{RE}=22.5%		TREATY C: XL AND λ_{RE}=10.8%	
T	EXP. RETURN (0,T) %	UES(T) / B_T ‰	EXP. RETURN (0,T) %	UES(T) / B_T ‰	EXP. RETURN (0,T) %	UES(T) / B_T ‰	EXP. RETURN (0,T) %	UES(T) / B_T ‰
1	9.97	0.0064	4.28	0.0026	9.16	0.0087	8.18	0.0000
2	20.97	0.0228	8.77	0.0112	19.11	0.0123	17.07	0.0005
3	33.05	0.0584	13.45	0.0304	30.11	0.0476	**26.84**	**0.0068**
4	46.44	0.1259	18.42	0.0934	42.28	0.1093	37.42	0.0288
5	61.12	0.2253	23.56	0.2375	55.37	0.2093	49.03	0.0761

Here mainly two different risk measures have been taken into account for these analyses: the finite time ruin probability and the unconditional expected shortfall.
As above mentioned in our numerical examples both risk measures drive to the same strategies and a similar measure of risk is estimated. This is mainly due to the fact that a single-line insurer is here regarded and the right tail of the amount of claim distribution X is regularly decreasing. The same two risk measures would likely drive to different strategies in case a multi-line insurer is analysed, expecially in case the aggregate claim distribution of X for all the lines would present multiple local modes on the right tail, with a general impact depending on the weight and the concentration of the claim distribution of each line of business.

8. FINAL COMMENTS.

As emphasized in this paper a reliable comparison of the results given by different reinsurance covers provided by the real market makes the insurer able to identify the most appropriate strategic planning. Moreover, Monte Carlo simulation technique could provide a useful insight of the whole risk process, with special advantage in cases of portfolios with a large skewness of the loss distribution, where the use of approximation formula is not reliable.

The risk theoretical model described is clearly a simplified version of the complex model to be taken into account, but here suitable analyses about primary insurance aspects have been preferred.
At this regard an appropriate analysis is highly recommended on investment and insurance long-term cycles, loss reserving run-off risk, stochastic investment return, dividends, taxation and ALM. These aspects will be the target of further researches in connection with the analysis of a multiline general insurer, where the combination of so many parameters and the correlation among more random variables should be summed up in the comprehensive risk profile of a general insurer. Moreover, suitable investigations are also needed when “financial” reinsurance programs are included in the available covers.
The real insurance world is clearly much more complicated, but this kind of approach, nowdays widespread in the actuarial literature under the umbrella term DFA (Dynamic Financial Analysis), is of real interest and it is promising of large improvements for a rational and coherent practical risk management in insurance.

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