ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Pricing life insurance contracts with early exercise features

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ARTICLE INFO

Article history:
Received 1 December 2007
Received in revised form 10 April 2008

MSC:
IE10
IE50
IB10

Keywords:
Insurance contracts
Surrender options
Least squares Monte Carlo method
American contingent claims

ABSTRACT

In this paper we describe an algorithm based on the Least Squares Monte Carlo method to price life insurance contracts embedding American options. We focus on equity-linked contracts with surrender options and terminal guarantees on benefits payable upon death, survival and surrender. The framework allows for randomness in mortality as well as stochastic volatility and jumps in financial risk factors. We provide numerical experiments demonstrating the performance of the algorithm in the context of multiple risk factors and exercise dates.

1. Introduction

The main characteristic of life insurance contracts is to provide benefits contingent on survival or death of individuals. While the provision of protection against the risk of death is certainly important, the savings component plays a crucial role in many life policies. In the last decades, the competition with alternative investment vehicles offered by the financial industry has generated substantial innovation in the design of life products and in the range of benefits provided. In particular, equity-linked policies have become more and more popular, offering policyholders exposure to financial indices as well as different ways to consolidate investment performance over time. While the introduction of more appealing (and exotic) guarantees has ensured a satisfactory take-up of insurance products, at the same time the management of life policies has become increasingly complex, requiring the proper understanding and analysis of integrated financial and insurance risks.

One of the most common options available in policies with a considerable savings component[1] is the possibility to exit (surrender) the contract before maturity and to receive a lump sum (surrender value) reflecting the insured's past contributions to the policy, minus any costs incurred by the company and possibly some charges. The idea is to boost sales by ensuring that the policyholder does not perceive insurance securities as an illiquid investment. On the other hand, surrenders are not welcomed by insurers, as they imply a reduction in the assets under management and may generate imbalances in the exposure to the mortality risk of remaining insureds (selective surrenders). An additional layer of risk is introduced when surrender values allow for minimum guarantees. The objective of the present paper is to study products embedding this type of option and guarantees.

From the point of view of asset pricing theory, surrender options are nothing else than American claims with a knock-out feature represented by the occurrence of death. In other words, a surrender option is an American put option written

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[1] An exception is represented by immediate annuities, because of high antiselection risk.

0377-0427/$ – see front matter
doi:10.1016/j.cam.2008.05.036

Please cite this article in press as: A.R. Bacinello, et al., Pricing life insurance contracts with early exercise features, Journal of Computational and Applied Mathematics (2008), doi:10.1016/j.cam.2008.05.036

on the residual value of the policy and canceled upon death, with a strike price given by the surrender value. A number of approaches to American option pricing have been proposed in the literature (see [13,15, and references therein]). The difficulty with insurance contracts is that realistic models pose formidable computational tasks for two main reasons. First, insurance contracts are usually very long term and offer great flexibility in early termination of the contract, meaning that we are faced with multiple exercise dates. Second, a realistic model should take into account at least the key drivers of rational surrender decisions, which in the equity-linked case may be quite involved. Incorporating irrational or exogenous factors as well as parameter uncertainty would make the task even more daunting.

Empirical evidence suggests that surrenders are driven by several factors including distribution channels, misselling, financial market conditions and deterioration/improvement of policyholders' health. In this paper, we focus on financial and demographic drivers, namely interest rate risk, investment performance and mortality risk. High interest rates as well as poor investment performance are usually associated with policyholders exiting the contract to enter more rewarding investment opportunities. This generates outflows that reduce assets under management and increase per-policy costs associated with insurance business. With regard to demographic factors, policyholders surrender policies when they perceive there is no need for the protection offered by the policy. On the other hand, policyholders do *not* surrender policies when protection is perceived valuable, even if they should or if the level of mortality protection is marginal in the overall design of the policy. This generates antiselection risk (selective surrenders) and can lead to dramatic changes in the demographic profile of an insurance portfolio. Mortality risk and surrender risk compound when sums at risk (death benefits minus reserves) are large. This aspect may be relevant even for equity-linked products, for example when death benefits are nominally fixed or provide substantial minimum guarantees. Irrespective of the specific risk factor considered, it is usually difficult to analyze all these risks simultaneously, as some of the benefits may be linked, others nominally fixed, and all of them may have substantial minimum guarantees.

Several simulation methods for pricing American options are available (see [15]), but they are usually not very effective in the presence of multiple state variables and several exercise dates. Since our objective is to take into account the risk factors described above and to allow for a range of real-world markets features (stochastic volatility, jumps in asset prices, randomness in mortality rates, etc.), we focus on the powerful Least Squares Monte Carlo (LSMC) approach, which was proposed in [11,18,24] in the context of purely financial American claims.

Surrender options have attracted the interest of many researchers. Starting with the seminal papers in [1,16,17], a number of studies have followed. Due to the high dimensionality of the problem, the vast majority of papers provide results in stylized situations. When moving to more realistic models, contributions become scarce. As an example, the introduction of mortality risk is carried out in only a few papers. For instance, we mention [3–5] in the context of binomial trees, [19,22] in the context of free boundary problems, [2,6] in the context of Monte Carlo simulation and the LSMC approach. We note that previous models typically use deterministic (or even constant) mortality rates. This is usually justified by adopting diversification arguments and invoking the law of large numbers for large enough insurance portfolios. However, the surrender decision is made by individual policyholders, who are faced with their own time of death only. This alone would justify the analysis of randomness in mortality rates. The issues of selective withdrawals and possibly large sums at risk make the case for random mortality even more compelling. Finally, we note that the application of the LSMC approach in the context of demographic risk requires care. This motivated the work in [6], which is further developed and extended in this paper.

The paper is structured as follows. Section 2 is devoted to the description of our valuation framework. In particular, we introduce the life insurance contracts of interest, define the dynamics of financial and demographic risk factors and finally present the valuation problem. In Section 3, we briefly describe the LSMC methodology and present our valuation algorithm. Section 4 offers some numerical results, while Section 5 concludes.

2. Valuation of equity linked endowments

2.1. The contract

Consider an individual aged x at time 0 when entering an endowment contract, i.e. a life policy with maturity $T > 0$ providing a lump sum benefit F_T^s at time T upon survival or a benefit F_t^d at time $t \in (0, T]$ in case t coincides with the individual's time of death, denoted by τ. If $F_T^s = 0$ the contract reduces to a term assurance policy, if $F_\tau^d = 0$ the contract reduces to a pure endowment. We will focus in particular on equity-linked products, meaning that (some of) the benefits are linked to the performance of a reference fund. Besides providing death and survival benefits, these policies may allow policyholders to exit the contract before maturity. If no payment is provided upon withdrawal, the policy is said to be lapsed. If instead a lump sum F_t^w is paid upon withdrawal at time t, the policy is said to be surrendered against provision of the *surrender value* F_t^w. Endowments can usually be surrendered, given the substantial savings component of the contract. On the other hand, term assurances are usually lapsed, since premiums and reserves are small and meant to provide pure protection.

Let us denote by $S \doteq (S_t)_{t \geq 0}$ the market value of the reference fund to which policy payments are linked. Benefits provided by equity-linked contracts typically embed minimum guarantees. A common example is represented by *terminal guarantees* of the form

$$F_t^e = F_0 \max\left(\frac{S_t}{S_0}, \exp(\kappa_e t)\right), \tag{2.1}$$

where $e = s, d, w$, depending on whether we consider survival, death or surrender benefits. In the above expression F_0 represents the initial value of the reference fund, which is financed by the policyholder's initial premium (net of any charges), while κ_e represents the minimum interest guaranteed on the different benefit payments.

In this paper, we limit our attention to the case of single premium policies. We refer the reader to [4] for considerations on multiple premium payments.

In expression (2.1) benefits depend only on the value of the reference fund at the relevant date. Some contracts are instead characterized by path-dependent guarantees. An example is represented by 'cliquet' guarantees, where guaranteed amounts are re-set regularly, possibly taking into account the performance of a reference fund at several past dates. For instance, we may consider payoffs defined by

$$F_t^e = F_0 \prod_{u=1}^{\lfloor t \rfloor} \max\left(1 + \frac{\eta}{u \wedge y} \sum_{j=1}^{u \wedge y} \left(\frac{S_{u-j+1}}{S_{u-j}} - 1\right), \exp(\kappa_e)\right), \tag{2.2}$$

meaning that the policyholder is awarded a proportion $\eta \in (0, 1]$ of the fund's performance smoothed over the last y years, subject to a minimum guarantee κ_e. Guarantees such as (2.2) are very common in *participating contracts*, i.e. policies where S represents the insurance company's investment portfolio rather than some external reference fund.

In this paper we focus on benefits of the type given in (2.1), although the analysis of path-dependent surrender options would require minor modifications.

Denote now by θ the time at which the policyholder decides to terminate the contract. Early termination can clearly occur if the individual is still alive and the policy is still in force. This poses no problem if $\theta < \tau \wedge T$, while we disregard surrender whenever $\theta \geq \tau \wedge T$. The time θ is in general a random variable whose law depends on the evolution of market and demographic conditions, which at any given time make the surrender value more or less attractive with respect to staying in the contract. We call therefore θ an exercise *policy*.

For given θ and fixed time t, we define the cumulated benefits paid by the contract up to that time by

$$G_t(\theta) \doteq F_T^s\, 1_{\tau > T, T \leq t \wedge \theta} + F_\tau^d 1_{\tau \leq t \wedge T \wedge \theta} + F_\theta^w 1_{\theta \leq t,\, \theta < \tau \wedge T}, \tag{2.3}$$

where the summands on the right hand side are contingent on the occurrence of three mutually exclusive events. Our objective is to introduce an arbitrage-free securities market and determine a surrender policy θ^* that is optimal for a rational policyholder.

2.2. Valuation framework

We take as given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, a filtered probability space supporting all sources of financial and demographic randomness. The filtration $\mathbb{F} \doteq (\mathcal{F}_t)_{t \geq 0}$ (satisfying the usual conditions of right continuity and completeness, and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$) represents the flow of information available to the insurer and the policyholder. It is natural to assume that both τ and θ are $\mathbb{F}$-stopping times, meaning that at any time t the information carried by $\mathcal{F}_t$ allows us to tell whether death or surrender have occurred or not by time t. The probability $\mathbb{Q}$ is a risk-neutral probability measure, meaning that under $\mathbb{Q}$ the market value of any security is given by the expected value of its cumulated dividends deflated at the risk-free rate. Under the assumption of frictionless securities markets, the existence of such $\mathbb{Q}$ is essentially guaranteed by the absence of arbitrage, a minimal requirement avoiding the availability of riskless gains at zero cost (see [13] for example).

Before introducing a model for the relevant risk factors, we give more structure to the information flow $\mathbb{F}$. We introduce the death indicator process $N_t \doteq 1_{\tau \leq t}$, which equals zero as long as the individual is alive and jumps to one at death. Denoting by $\mathbb{H}$ the filtration generated by $(N_t)_{t \geq 0}$, we assume that $\mathbb{F} \doteq \mathbb{G} \vee \mathbb{H}$ for some filtration $\mathbb{G}$ not including $\mathbb{H}$ and such that $\mathcal{G}_0$ is trivial. The intuition is that $\mathbb{G}$ carries all relevant information about demographic and financial factors (in particular, security prices and likelihood of death), but does not yield knowledge of occurrence of τ. A possible way of defining the arrival of death is by setting

$$\tau \doteq \inf\{t : \Gamma_t > \xi\}, \tag{2.4}$$

with $(\Gamma_t)_{t \geq 0}$ a $\mathbb{G}$-adapted nondecreasing process and ξ a random variable independent of $\mathcal{G}_\infty$ and exponentially distributed with parameter one. If Γ can be expressed as $\Gamma_t = \int_0^t \mu_s \mathrm{d}s$ for all t and some $\mathbb{G}$-predictable nonnegative process μ, then construction (2.4) is equivalent to the so called conditionally Poisson setup. This means that, conditionally on $\mathcal{G}_\infty$ and under the measure $\mathbb{Q}$, τ is the first jump time of a Poisson inhomogeneous process with intensity $(\mu_t)_{t \geq 0}$. This setup is appealing because it generalizes the usual formulas employed in demography, insurance or economics to a stochastic framework (e.g., [9]). The setting also ensures that any $\mathbb{G}$-martingale is an $\mathbb{F}$-martingale,[2] yields considerable simplifications in pricing formulas, as will be explained in Section 2.4.

[2] To see this, note that construction (2.4) implies $\mathbb{Q}(\tau > t | \mathcal{G}_\infty) = \mathbb{Q}(\tau > t | \mathcal{G}_t) = \exp(-\Gamma_t)$. In other words, for each $t \geq 0$, $\mathcal{H}_t$ and $\mathcal{G}_\infty$ are conditionally independent, given $\mathcal{G}_t$. For any $\mathbb{G}$-martingale $(M_s)_{s \geq 0}$ and $T \geq t \geq 0$ we can then write $E^{\mathbb{Q}}[M_T | \mathcal{F}_t] = E^{\mathbb{Q}}[M_T | \mathcal{F}_t] = M_t$.

2.3. Financial and demographic risk factors

As explained in the introduction, empirical evidence on insurance markets suggests that surrender decisions are driven by several factors including distribution channels, misselling, financial market conditions and deterioration or improvement of policyholders' health. Here, we focus on financial and demographic drivers of surrender decisions, in particular on interest rate risk, stock market performance and mortality risk.

We model the term structure of interest rates by a standard Cox–Ingersoll–Ross model, a square-root process featuring mean reverting and nonnegative rates. Its dynamics is given by

$$\mathrm{d}r_t = \zeta_r(\delta_r - r_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}Z_t^r,$$

with $\zeta_r, \delta_r, \sigma_r > 0$ and Z^r a standard Brownian motion. For the market value of the reference fund, we consider the stochastic exponential $S = \exp(Y)$, where Y evolves according to

$$\mathrm{d}Y_t = \left(r_t - \frac{1}{2}K_t - \lambda_Y\mu_Y\right)\mathrm{d}t + \sqrt{K_t}\left(\rho_{SK}\mathrm{d}Z_t^K + \rho_{Sr}\mathrm{d}Z_t^r + \sqrt{1-\rho_{SK}^2-\rho_{Sr}^2}\mathrm{d}Z_t^S\right) + \mathrm{d}J_t^Y.$$

The component J^Y is a compound Poisson process with jump arrival rate $\lambda_Y > 0$ and lognormally distributed jump sizes with mean μ_Y and standard deviation $\sigma_Y > 0$. The correlation coefficients ρ_{SK} and ρ_{Sr} satisfy $\rho_{SK}^2 + \rho_{Sr}^2 \leq 1$. The factor K generates stochastic volatility (indeed, K is the square of the instantaneous non-jump volatility of S) with mean-reverting dynamics

$$\mathrm{d}K_t = \zeta_K(\delta_K - K_t)\mathrm{d}t + \sigma_K\sqrt{K_t}\mathrm{d}Z_t^K,$$

where $\zeta_K, \delta_K, \sigma_K > 0$. We take the process (Z^r, Z^S, Z^K) to be a three dimensional Brownian motion independent of the pure jump process J^Y. The above formulation is essentially the one studied by [8], who show that it is quite effective in reproducing the price behavior of equity derivatives.

For the intensity of mortality, we take the left continuous version of the process

$$\mathrm{d}\mu_t = \zeta_\mu(m(t) - \mu_t)\mathrm{d}t + \sigma_\mu\sqrt{\mu_t}\mathrm{d}Z_t^\mu + \mathrm{d}J_t^\mu, \tag{2.5}$$

where $m(\cdot), \zeta_\mu, \sigma_\mu > 0$, Z^μ is a standard Brownian motion and J^μ is a compound Poisson process independent of Z^μ, with jump arrival rate $\lambda_\mu \geq 0$ and exponential jumps of mean $\gamma_\mu > 0$. Finally, the couple (Z^μ, J^μ) is assumed to be independent of (Z^r, Z^S, Z^K, J^Y), so that financial and demographic factors are independent.

The choice of risk factors outlined in this section is aimed at providing a realistic setup to assess the performance of the LSMC methodology. Alternative dynamics could be analyzed, but we think that the current setup represents a valid test, given the number of factors and the presence of multiple exercise dates. To conclude, we set $X = (N, \mu, r, Y, K)$ and denote by $\widehat{X} = (\mu, r, Y, K)$ the 'reduced' state variable process. We then take $\mathbb{F}$ to be the filtration generated by X and $\mathbb{G}$ the one generated by $\widehat{X}$. We observe that both X and $\widehat{X}$ are affine processes. As a result, when implementing the algorithm, we have at least a benchmark for the case of European guarantees, since they can be readily priced in the affine setting (see [10]).

2.4. Valuation

Our financial market is characterized by the investment fund S introduced in the previous section and a money market account yielding the instantaneous risk-free rate $(r_t)_{t\geq 0}$. For each $t \geq 0$, $B_t = \exp(\int_0^t r_s\mathrm{d}s)$ formalizes the proceeds from investing one unit of money at time 0 in risk-free deposits and rolling over the proceeds until time t. Consider now the insurance contract introduced in Section 2.1, where we assume that the process F^e is $\mathbb{G}$-predictable for $e = s, d, w$: this always holds when S is defined as in Section 2.3 and we consider the left-continuous versions of (2.1) or (2.2). In the present framework and consistently with the classical treatment of American options, it is natural to assume that any exercise policy θ is an $\mathbb{F}$-stopping time, meaning that the surrender decision is based on the entire information available over time.

Suppose that the contract is terminated at the random time θ: under no arbitrage, the time-t value $V_t(\theta)$ of the contract is then given by the usual risk-neutral formula

$$V_t(\theta) = B_t\, E^{\mathbb{Q}}\left[\int_t^\infty B_u^{-1}\mathrm{d}G_u(\theta)\middle|\mathcal{F}_t\right], \tag{2.6}$$

where $G_u(\theta)$ is given by (2.3) and represents the cumulated benefits provided by the contract up to time $u \geq t$ if the exercise policy θ is adopted.

Exploiting the structure of the filtration $\mathbb{F}$, in [21, p. 370] we know that every $\mathbb{F}$-stopping time θ coincides with a $\mathbb{G}$-stopping time $\bar{\theta}$ up to time τ. Assuming that τ coincides with the first jump of a conditionally Poisson process with intensity μ, expression (2.6) reduces to (e.g., [14])

$$V_t(\theta) = 1_{\tau>t}\, \widehat{B}_t\, E^{\mathbb{Q}}\left[\int_t^\infty \widehat{B}_u^{-1}\mathrm{d}\widehat{G}_u(\bar{\theta})\middle|\mathcal{G}_t\right], \tag{2.7}$$

with $\widehat{B}_\cdot = \exp(\int_0^\cdot (r_s + \mu_s)\mathrm{d}s)$ and

$$\widehat{G}_u(\theta) = F_T^s\, 1_{T \le u \wedge \theta} + \int_0^u F_s^d\, \mu_s\, 1_{s \le T \wedge \theta}\, \mathrm{d}s + F_\theta^w\, 1_{\theta \le u, \theta < T}.$$

As a result, everything works as in the absence of the random time of death, provided we replace both the money market account and the death benefit with their 'mortality risk-adjusted counterparts' $(\widehat{B}_s)_{s \ge t}$ and $(F_s^d \mu_s)_{s \ge t}$.

Let us now focus on the pricing of the contract at inception. Denoting by $\mathcal{T}_{\mathbb{F}}$ ($\mathcal{T}_{\mathbb{G}}$) the set of finite valued $\mathbb{F}$-stopping times ($\mathbb{G}$-stopping times), the initial price is given by the solution of the optimal stopping problem

$$V_0^* = V_0(\theta^*) = \sup_{\theta \in \mathcal{T}_{\mathbb{F}}} V_0(\theta) = \sup_{\bar{\theta} \in \mathcal{T}_{\mathbb{G}}} V_0(\bar{\theta}) = \sup_{\bar{\theta} \in \mathcal{T}_{\mathbb{G}}, \bar{\theta} \le \tau} V_0(\bar{\theta}), \tag{2.8}$$

where the last two equalities follow again by [21, p. 370] and the fact that $V_0(\bar{\theta}) = V_0(\bar{\theta} \wedge \tau)$. Since θ and $\bar{\theta}$ coincide up to τ, as seen from contract inception, from now on we simply use the notation θ when dealing with (2.8). A solution θ^* to problem (2.8) is called a rational exercise policy, in the sense that it maximizes the initial arbitrage-free value of the resulting claim. While (2.8) can be justified by replication arguments when markets are complete, the case of incomplete markets is more delicate (e.g., [13]). This is exactly our situation, even if we disregard mortality risk, because our financial market is incomplete and perfect replication is in general not possible. We do not expand on this here and simply employ (2.8) under our given fixed risk-neutral probability measure $\mathbb{Q}$.

While expression (2.7) is extremely appealing from the point of view of interpretation, it can be computationally more intensive than (2.6), as explained in [7]. This is why for the pricing algorithm described in the next section we use the last equality in (2.8), but work directly with expression (2.6).

2.5. Extension to exogenous surrender factors

The model described so far can be extended to include exogenous surrender decisions, where by exogenous we mean that the decision to withdraw from the contract may be triggered by something different from continuation values falling below surrender benefits. A natural way of capturing this possibility is to introduce another stopping time ν with $\mathbb{G}$-predictable intensity $(\phi_t)_{t \ge 0}$, where we emphasize that $\mathbb{G}$ could be generated by non-financial and non-demographic factors. Inspection of expression (2.8) shows that the optimal stopping problem can be modified by simply replacing τ with $\tau \wedge \nu$. Since τ and ν admit intensities μ and ϕ, the stopping time $\tau \wedge \nu$ admits the intensity $\mu + \phi$. If $\tau \wedge \nu$, rather than τ, is assumed to satisfy the conditions described in Section 2.2, then the valuation framework can be applied with no substantial changes, apart from replacing μ with $\mu + \phi$. Similarly, the algorithm described in the next section can be used with no substantial modification.

3. The algorithm

The LSMC approach is based on the joint use of Monte Carlo simulation and Least Squares regression in Markovian environments. The method first requires discretization of the time dimension, in order to replace the original optimal stopping problem (2.8) with its discretized version along a time grid $\mathbb{T}$. Denoting by n the number of periods in which we divide the interval $[0, T]$ and setting $t_i = \frac{i}{n}T$ for $i = 0, \ldots, n$, we obtain a time grid $\mathbb{T} = \{t_0, \ldots, t_n\}$ and a discretized stopping problem

$$\sup_{\theta \in \mathcal{T}_{\mathbb{F},\mathbb{T}}} E^{\mathbb{Q}}\left[g_\theta\right], \tag{3.1}$$

with $\mathcal{T}_{\mathbb{F},\mathbb{T}}$ denoting the family of $\mathbb{T}$-valued $\mathbb{F}$-stopping times and

$$g_t \doteq \int_0^t B_0^{-1} \mathrm{d}G_s(\theta),$$

where $G_s(\theta)$ is defined by (2.3). As is common when dealing with American options, one can introduce the Snell envelope of $(g_t)_{t \ge 0}$ and apply the dynamic programming principle to develop a backward procedure involving a comparison, at each time step, between the payoff provided by the surrender option and the *continuation value* (i.e., the reward from not exercising). The LSMC method looks at such procedure in terms of optimal stopping times, in the sense that an optimal policy $\theta^* = \theta_0^*$ is computed according to the backward algorithm

$$\begin{cases} \theta_n^* = t_n = T \\ \theta_j^* = t_j 1_{g_{t_j} > U_j} + \theta_{j+1}^* 1_{g_{t_j} \le U_j} & \text{for } j = n-1, \ldots, 0, \end{cases}$$

where $U_j \doteq E^{\mathbb{Q}}[g_{\theta_{j+1}^*} | \mathcal{F}_{t_j}]$.

Since we work in a Markovian environment, we can write $U_j = E^{\mathbb{Q}}[g_{\theta_{j+1}^*} | X_{t_j}] = u(t_j, X_{t_j})$, for some Borel functions $u(t, \cdot)$, $t \in \mathbb{T}$. A second approximation involves replacing each $u(t_j, X_{t_j})$ with the orthogonal projection from $L^2(\Omega)$ onto the H-dimensional vector space generated by a finite set of functions taken from a suitable basis $\{e_1, \ldots, e_H, \ldots\}$. For fixed H

and each j, we define such projection by $\widetilde{u}(t_j, X_{t_j}) \doteq \beta_j^* \cdot e(X_{t_j})$, where $e(\cdot) \doteq (e_1(\cdot), \dots, e_H(\cdot))'$ and $\beta_j^* \doteq (\beta_j^{*1}, \dots, \beta_j^{*H})'$ is the optimal vector obtained by least squares regression. In particular, we simulate the state variable process X over the time grid $\mathbb{T}$ (or over a finer grid) and then set

$$\beta_j^* = \arg\min_{\beta_j \in \mathbb{R}^H} \sum_{m=1}^{M} \left(g^m_{\theta^*_{j+1}} - \beta_j \cdot e(X^m_{t_j}) \right)^2, \tag{3.2}$$

where M denotes the number of simulations and $X^m_{t_j}$, $g^m_{t_j}$ denote the simulated values of X_{t_j} and g_{t_j} in the m-th simulation, for $m = 1, \dots, M$. Convergence results for the LSMC algorithm are provided in [12], while numerical results for several choices of basis functions are reported in [20,23]. We now show how to apply the LSMC method to the setup of Section 2.

3.1. Algorithm

Assume that M simulated paths have been generated for the reduced state variables process $\widehat{X}$ and a unit exponential random variable. For each simulation m ($m = 1, \dots, M$), we let ξ^m denote the simulated value of the exponential random variable and by $(\mu^m_t)_{t\in\mathbb{T}}$ a simulated path of the stochastic intensity given in (2.5). According to (2.4), the simulated time of death is obtained by setting

$$\tau^m = \min\{t \in \mathbb{T} : \Gamma^m_t > \xi^m\},$$

where Γ^m_t represents the (approximated) value of the integral $\int_0^t \mu^m_s \mathrm{d}s$. If the insured survives at maturity T (the set above is empty), we set $\tau^m = +\infty$ by convention. Of course, here and in the sequel, one can simulate random processes over a grid finer than $\mathbb{T}$ to improve the approximations. For each $t \in \mathbb{T}$ such that $t \le \tau^m$, we denote by r^m_t, K^m_t and S^m_t the simulated values of the short rate, stochastic volatility and reference fund. We can then compute the simulated discount factors, denoted by $v^m_{t,s} \doteq B^m_t (B^m_s)^{-1}$ (with $t < s$ and $t, s \in \mathbb{T}$) and the simulated benefits payable on death ($F^{d,m}_t$), survival ($F^{s,m}_t$) and surrender ($F^{w,m}_t$). The valuation algorithm requires execution of the following steps:

STEP 1. (*Initialization*)
For $m = 1, \dots, M$, if $\tau^m \le T$ set $\theta^{*,m} = \tau^m$ and $P^m_{\theta^{*,m}} = F^{d,m}_{\theta^{*,m}}$, otherwise set $\theta^{*,m} = T$ and $P^m_{\theta^{*,m}} = F^{s,m}_{\theta^{*,m}}$.

STEP 2. (*Backward iteration*)
For $j = n-1, n-2, \dots, 1$:
(1) (*Continuation values*) Let $I_j = \{1 \le m \le M : \tau^m > t_j\}$ and, for each $m \in I_j$, set $C^m_{t_j} = P^m_{\theta^{*,m}}\, v^m_{t_j,\theta^{*,m}}$
(2) (*Regression*) Regress the continuation values $C_{t_j} = (C^m_{t_j})_{m\in I_j}$ against $(e(X^m_{t_j}))_{m\in I_j}$ to obtain $\widetilde{C}^m_{t_j} = \beta_j^* \cdot e(X^m_{t_j})$ for each $m \in I_j$. If $F^{w,m}_{t_j} > \widetilde{C}^m_{t_j}$ then set $\theta^{*,m} = t_j$ and $P^m_{t_j} = F^{w,m}_{t_j}$.

STEP 3. (*Initial value*)
Compute the single premium of the contract by

$$V_0^* = \frac{1}{M} \sum_{m=1}^{M} P^m_{\theta^{*,m}}\, v^m_{0,\theta^{*,m}}.$$

The introduction of exogenous factors in the surrender decision, as outlined in Section 2.5, can be done as follows. We essentially need to focus on $\iota \doteq (\tau \wedge \nu)$ rather than τ, and replace μ with $\mu + \phi$. For example, the simulated realization ι^m can be computed by using:

$$\iota^m = \min\left\{ t \in \mathbb{T} : \int_0^t (\mu^m_s + \phi^m_s)\mathrm{d}s > \xi^m \right\}. \tag{3.3}$$

For the initialization in Step 1, we need of course to distinguish between death and surrender in case $\iota^m \le T$, unless death and surrender benefits coincide. This can be done by drawing a Bernoulli for the conditional death event, with probability

$$\mathbb{Q}(\tau = \iota | \iota = \iota^m) = \frac{E^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^{\iota^m} (\mu_s + \phi_s)\mathrm{d}s} \mu_{\iota^m} \right]}{E^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^{\iota^m} (\mu_s + \phi_s)\mathrm{d}s} (\mu_{\iota^m} + \phi_{\iota^m}) \right]}, \tag{3.4}$$

where we note that both numerator and denominator can be computed in closed form in the affine setting, or approximated numerically using the already available paths μ^m and ϕ^m. Alternatively, one could draw two unit exponentials ξ^m_τ and ξ^m_ν and simulate τ and ν individually instead of using (3.3) and (3.4).

The simplest way to obtain the initial value of the surrender option (rather than the entire contract) is to compute the initial value V_0 of the European version of the contract, and then recover the surrender option price from the difference $V_0^* - V_0$. When V_0 is not available in closed form, we can compute it by executing only STEP 1 and STEP 3 of our algorithm. The next section offers numerical results on the implementation of the above procedure.

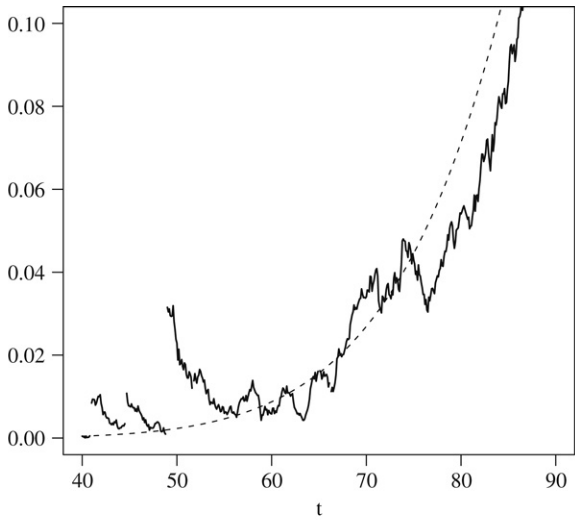

Fig. 1. Solid line: sample path of μ. Dashed line: Weibull intensity m.

Table 1
Parameters used in the simulation

	r	K	S	μ
BDS = 1.00	$r_0 = 0.05$	$K_0 = 0.04$	$S_0 = 100.00$	$\mu_0 = m(0)$
FDS = 0.01	$\zeta_r = 0.60$	$\zeta_K = 1.50$	$\rho_{SK} = -0.70$	$\zeta_\mu = 0.50$
	$\delta_r = 0.05$	$\delta_K = 0.04$	$\rho_{Sr} = 0.00$	$\sigma_\mu = 0.03$
	$\sigma_r = 0.03$	$\sigma_K = 0.40$	$\lambda_Y = 0.50$	$\lambda_\mu = 0.10$
			$\mu_Y = 0.00$	$\gamma_\mu = 0.01$
			$\sigma_Y = 0.07$	$c_1 = 83.70$
				$c_2 = 8.30$
				$x = 40$

Table 2
Value of the contract

	κ (%)			κ_w (%)
	0	2	4	
	113.556 (0.031)	115.381 (0.033)	123.087 (0.033)	0
	117.223 (0.031)	117.551 (0.031)	123.291 (0.033)	2
	123.687 (0.031)	123.727 (0.031)	124.507 (0.032)	4
	137.130 (0.031)	137.327 (0.030)	137.710 (0.030)	6
V_0	107.185 (0.047)	112.675 (0.045)	122.901 (0.041)	

4. Numerical examples

We consider the single premium equity-linked endowment described in Section 2.1. The reference insured is a male aged $x = 40$ at time 0. The contract has maturity $T = 15$ years and provides terminal guarantees on survival, death and surrender benefits, as defined in (2.1). We apply the valuation algorithm with polynomial basis functions of order 3, yielding $H = 34$. To replace the American claim with a Bermudan claim, we discretize the time dimension by using a time step (in years) which we call Backward Discretization Step (BDS). To simulate the reduced state variable process $\widehat{X}$, we employ a time grid finer than $\mathbb{T}$ and call Forward Discretization Step (FDS) the length in years of each time interval in the finer grid. All the relevant parameters used for our simulations are reported in Table 1.

With regard to mortality dynamics, we note that the function m in (2.5) is obtained by fitting a Weibull intensity, given by $m(t) = c_1^{-c_2} c_2 (x+t)^{c_2-1}$ (with $c_1 > 0$, $c_2 > 1$), to the survival probabilities implied by table SIM2001, commonly used in the Italian endowments market. A sample path of μ is reproduced in Fig. 1.

We have considered different values for the minimum rates guaranteed upon both death and survival ($\kappa \doteq \kappa_d = \kappa_s$), as well for the surrender guarantee (κ_w). Of course, the price of the European contract does not depend on κ_w. The case of $k_w \leq k$ is what one encounters in practice, for otherwise contract design would favor withdrawals. We consider the case of $k_w > k$ for numerical purposes only. We ran 19 000 simulations with 140 different seeds to obtain the results reported in Tables 2 and 3. The last row of Table 2 reports the value of the European contract (V_0), while the other rows report the values

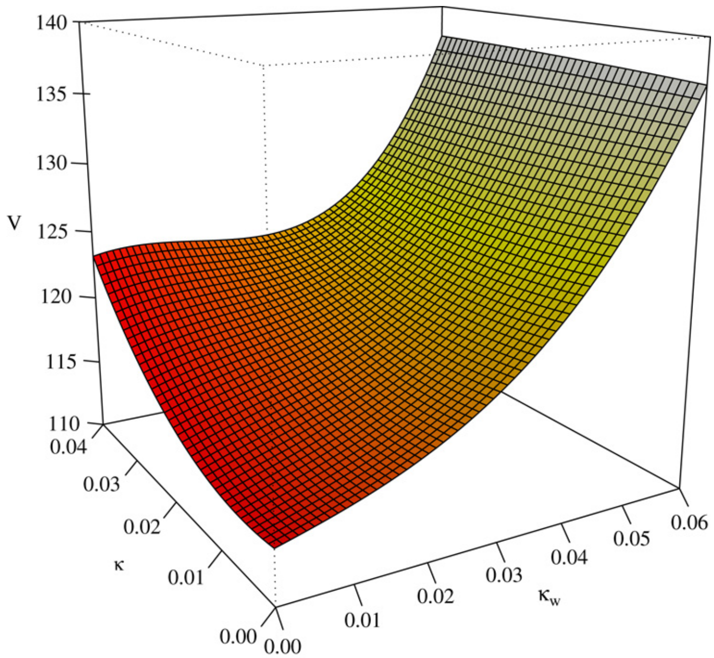

Fig. 2. Value of the American contract.

Table 3
Surrender option value

	κ (%)			κ_w (%)
	0	2	4	
	6.372	2.706	0.186	0
	10.038	4.876	0.390	2
	16.503	11.052	1.606	4
	29.945	24.652	14.809	6

Table 4
Surrender option values and percentage changes of the higher mortality model with respect to the base case of Table 3

	κ (%)			κ_w (%)
	0	2	4	
	6.402 (+0.47%)	2.950 (+9.02%)	0.273 (+46.77%)	0
	9.884 (−1.53%)	4.994 (+2.42%)	0.407 (+4.35%)	2
	15.968 (−3.24%)	10.856 (−1.77%)	1.639 (+2.08%)	4
	27.980 (−6.56%)	23.105 (−6.27%)	14.148 (−4.46%)	6

of the American contract with standard errors in parenthesis. Finally, Table 3 reports the values for the surrender option, while Fig. 2 depicts the value of the American contract against both κ and κ_w.

As expected, the value of the European contract is increasing with the minimum interest rate guaranteed upon death or survival, κ. The value of the American contract is increasing with both κ and the minimum interest rate guaranteed upon surrender, κ_w. The value of the surrender option instead increases with κ_w and decreases with κ. The option value is negligible when κ is large relative to κ_w, because stepping out of the contract is then less attractive. [7] propose an alternative algorithm that can be used as a benchmark for our case and indeed confirms the validity of our results.

To understand the impact of mortality on our results, we compute surrender option values for a mortality model yielding a 9.67% decrease in expected lifetime (from 38.79 to 35.04 years). This is obtained by raising instantaneous volatility to $\sigma_\mu = 0.10$ and mean jump size to $\gamma_\mu = 0.04$. We report the results in Table 4, together with the percentage changes with respect to the base case of Table 3. We see that surrender options become more valuable in the higher mortality case as long as $\kappa \geq \kappa_w$. To understand why, we note that while higher mortality involves a lower probability of receiving payments upon survival (i.e., surrender and maturity benefits), at the same time it can be shown to increase the overall value of our standard endowment contracts. As a result, there are two effects at play when considering the optimal stopping problem (3.1): on the one hand, higher mortality increases the argument of the supremum, on the other hand, it reduces the probability of being able to exercise before death. When $\kappa < \kappa_w$, surrender guarantees are more valuable relative to death and survival

Please cite this article in press as: A.R. Bacinello, et al., Pricing life insurance contracts with early exercise features, Journal of Computational and Applied Mathematics (2008), doi:10.1016/j.cam.2008.05.036

guarantees, but the lower probability of exercising before death generates a decrease in option values. When instead $\kappa \geq \kappa_w$, surrender benefits are less valuable relative to death and survival benefits, but the increase in value for the stopped contract (induced by higher mortality) makes option values increase with respect to the base case.

5. Conclusions

In this paper we have described an algorithm for the valuation of a life insurance contract embedding surrender options based on the Least Squares Monte Carlo method. The approach is effective in that we can avoid imposing rigid simplifying assumptions on the valuation model. We have then shown how to implement the model in the context of equity-linked endowment contracts with minimum guarantees on death, surrender and survival benefits. We have considered different sources of uncertainty, such as random interest rates, stochastic volatility and jumps in asset prices, as well as random mortality rates. Finally, we have performed numerical experiments demonstrating the performance of the approach. Objectives for future research are the detailed analysis of the implications of market incompleteness and the introduction of policyholders' subjectivity in the surrender optimization problem.

Acknowledgements

The authors thank an anonymous referee for thoughtful comments and suggestions. The authors are solely responsible for any errors. The authors gratefully acknowledge financial support from the Italian Ministry of University and Research (MIUR) and the University of Trieste.

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