A Full Monte Carlo Approach to the Valuation of the Surrender Option Embedded in Life Insurance Contracts

Anna Rita Bacinello*

Dipartimento di Matematica Applicata alle Scienze Economiche, Statistiche ed Attuariali "Bruno de Finetti", Università degli Studi di Trieste
Piazzale Europa, 1
34127 Trieste, Italia
E-mail: bacinel@units.it

Final version published in *Mathematical and Statistical Methods for Insurance and Finance*, C. Perna and M. Sibillo (eds.), 2008, Springer, p. 19-26.

Summary. In this paper we extend the Least Squares Monte Carlo approach proposed by Longstaff and Schwartz for the valuation of American-style contingent-claims to the case of life insurance contracts. These contracts, in fact, often embed an American-style option, called surrender option, that entitles its owner to early terminate the contract and receive a cash amount, called surrender value. The additional complication arising in life insurance policies with respect to purely financial American contracts is that there is not a fixed date within which the option can be exercised, since its "maturity" is driven by mortality factors. This complication has been handled by very few papers, often at the cost of excessively simplified valuation frameworks. Then the aim of this contribution, that is not a specific valuation model but a methodological approach, is to allow a full exploitation of the flexibility inborn in Monte Carlo and quasi-Monte Carlo methods in order to deal with more realistic valuation frameworks.

Key words: surrender option, Least Squares Monte Carlo approach.

1 Introduction

The surrender option embedded in several types of life insurance contracts gives the policyholder the right to early terminate the contract, before its natural termination (that is typically *death* or *maturity*), and to receive a cash amount, called *surrender*

* Financial support from Cofinanziamento MIUR on "Valutazioni nelle assicurazioni vita e nel settore previdenziale: metodologie attuariali, economiche e finanziarie" is gratefully acknowledged.

value. It is a non-standard *knock-out* American put option written on the residual contract, with exercise price given by the surrender value. The knock-out feature is implied by the fact that this option can be exercised only if the insured is still alive, hence it expires in case of death ("knock-out event"). Then the additional complication arising in life insurance policies with respect to standard American options is that there is not a fixed date within which the option can be exercised, since its "maturity" is driven by mortality factors. Moreover, the value of the residual contract depends both on mortality and on financial uncertainty and, even if pooling arguments can be applied in order to hedge the mortality risk, it is not possible, in the valuation, to keep separate these two sources of uncertainty because there is a continuous interaction between them.

The literature concerning the valuation of the surrender option in a contingent-claims framework is not very abundant, and most of the papers on this subject deal with purely financial contracts, without mortality risk, applying them the results on American options. There are only very few exceptions that deal with actual life insurance contracts, characterized by both financial and mortality risk. However the complexity of the problem involved in this case often forces the assumption of excessively simplified valuation frameworks (deterministic interest rates, deterministic volatility for reference portfolios, deterministic mortality trends).

The valuation approaches followed to tackle the problem can be essentially classified in three categories:

1. binomial/multinomial trees (see [Bac03a], [Bac03b], [Bac05], [Van03a], [Van03b]);
2. Partial Differential Equations with free boundaries (see [SX05]);
3. Monte Carlo simulation (see [AC03], [BDF06]).

In particular, the papers [AC03] and [BDF06] combine the Least Squares Monte Carlo approach (LSM henceforth) proposed by [LS01] for the valuation of *purely-financial* American-style contingent-claims with the approach proposed by [Bac03a] and [Bac03b] to manage the mortality risk in the valuation of the surrender option. More in detail, they follow the LSM approach only to handle the financial uncertainty (stochastic interest rates, stochastic evolution of reference portfolios), but resort to "analytic" tools for the mortality one. Then the two sources of uncertainty are not treated in the same way and the mortality uncertainty does not enter either the simulation process or the definition of the stochastic discounted cash-flow of the contract.

This paper extends instead the LSM approach to the case of life insurance contracts in a very natural way, according to which the mortality uncertainty is treated exactly as the financial one and is part of the whole LSM mechanism. The aim of this contribution, that is not a specific valuation model but a methodological approach, is to allow a full exploitation of the flexibility inborn in Monte Carlo and quasi-Monte Carlo methods, in order to deal with very realistic valuation frameworks including, e.g., stochastic interest rates, stochastic volatility, jumps, stochastic mortality trends, ...

The paper is structured as follows. In Section 2 we present our notation and assumptions, in Section 3 we describe the valuation approach and in Section 4 we discuss about the numerical accuracy of its results. Finally, Section 5 concludes the paper.

2 Notation and assumptions

Consider an endowment policy issued at time t_0 and maturing at time t_N or, alternatively, a whole-life assurance policy issued at time t_0. In both cases assume that the policy is paid by a single premium at issuance, denoted by U. Assume moreover that the endowment policy can be surrendered at the discrete dates $t_1, t_2, ..., t_{N-1}$ before maturity, if the insured is still alive. Similarly, also the whole-life assurance policy can be surrendered at the discrete dates $t_n, n = 1, 2, ...$, belonging to a given set and before death of the insured. In case of death between times t_{n-1} and $t_n, n = 1, 2, ...$, the benefit is assumed to be paid at the end of the interval, that is at time t_n. Of course, if we consider an endowment policy and the insured is still alive at maturity t_N, the benefit is paid at maturity.

Observe that, to avoid adverse selection, the surrender option is usually offered only when a life insurance contract provides benefits with certainty (even if their amount and/or the time at which they are due are uncertain). That is why we limit ourselves to consider these two types of contracts, where the benefit is due with certainty, soon or later.

Assume that the contract under scrutiny is a unit-linked or a participating policy characterized, as usual, by a rather high level of financial risk. The benefit and the surrender value can then be linked to the value or to the performance of a reference portfolio, with possible minimum guarantees, or upper bounds, or other. Here it is not important to specify their structure. We only denote by $F_{t_n}, n = 0, 1, 2, ...$, the value of the reference portfolio at time t_n and by $R_{t_n}, n = 1, 2, ...$, the surrender value paid in case of surrender at time t_n. Then the "payoff" of the contract is given by the benefit, at death or maturity, or the surrender value, in case of surrender.

3 The valuation approach

In this section we describe our valuation approach. This approach concerns the whole contract, including the surrender option. Its output is the *fair value* of the contract at time 0, that is also the fair single premium to require for it. Note that with a drastic simplification, that we briefly describe at the end of the section, the same procedure provides the single premium for the corresponding European version of the contract, that is without surrender option. Then, if one is interested to separately value this option, the simplest way to do it is to compute the difference between the time 0 value of the whole contract and that of its European version.

The valuation procedure can be schematized in the following steps.

Step 1. Generate a certain number, say H, of (independent) simulated values of the remaining lifetime of the insured. Then, with reference to the h-th iteration ($h = 1, 2, ..., H$):
- let $T^{(h)}$ denote the simulated time at which the benefit would be due if the surrender option were never exercised;
- produce a simulated path of the stochastic term-structure of spot interest rates for any possible maturity up to time $T^{(h)}$;
- produce a simulated path of the reference portfolio at times $t_1, t_2, ..., T^{(h)}$.

Remarks: All simulations must be performed under a *risk-neutral* measure Q, taking into account all possible dependencies. For instance, a path for the volatility of zero-coupon bond prices and of the reference portfolio has also to be simulated if these volatilities are assumed to be stochastic. Moreover, if we assume a diffusion process to describe the value of the reference portfolio, its drift under the risk-neutral measure is given by the instantaneous spot rate, and so on. Similarly, if future mortality trends are assumed to be stochastic, the simulation of the remaining lifetime of the insured previously requires the simulation of all variables on which it depends such as, e.g., the instantaneous force of mortality.

Step 2. Let $t_{max} = \max \left\{ T^{(h)} : h = 1, 2, ..., H \right\}$. Then, with reference to all iterations (h) such that $T^{(h)} = t_{max}$, determine the "final" payoff of the contract, given by the simulated benefit at death or maturity, and denote it by $P^{(h)}_{t_{max}}$.

Remark: If we are dealing with an endowment policy, it is very likely $t_{max} = t_N$.

Step 3. Let $n = max - 1, max - 2, ..., 1$.

- With reference to all iterations (h) such that $T^{(h)} = t_n$, let $P^{(h)}_{t_n}$ be given by the corresponding simulated benefit.
- With reference to all iterations (h) such that $T^{(h)} > t_n$, use the Least Squares method to estimate

$$E^Q_{t_n} \left[\sum_{j:\, t_n < t_j \le T} P_{t_j} v \left(t_n, t_j \right) \right],$$

where $E^Q_{t_n}$ denotes expectation, under the (chosen) risk-neutral measure Q, conditioned to the information available up to time t_n and to the event that the contract is still in force at this time (i.e., insured still alive and contract not surrendered yet), T denotes the stochastic date at which the benefit would be due, P_{t_j} denotes the stochastic future payoff of the contract at time t_j, $v \left(t_n, t_j \right)$ denotes the stochastic discount factor from t_j to t_n at the riskless rate.

Then, denoting by $f^{(h)}_{t_n}$ the estimated conditional expectation, compare it with the corresponding simulated surrender value $R^{(h)}_{t_n}$:

if $R^{(h)}_{t_n} \le f^{(h)}_{t_n}$, let $P^{(h)}_{t_n} = 0$ and do not change the future payoffs $P^{(h)}_{t_j}, j > n$;

if $R^{(h)}_{t_n} > f^{(h)}_{t_n}$, let $P^{(h)}_{t_n} = R^{(h)}_{t_n}$ and $P^{(h)}_{t_j} = 0$ for any $j > n$.

Remarks: This step requires to choose the basis functions whose linear combination defines the regression function, as well as all relevant variables (e.g., the current value of the reference portfolio, the current spot rate, the current force of mortality, ..., and their past values if the model assumed is not Markovian). Note that, inside the Q-expectation, we have the sum of all future payoffs. Actually there is one and only one non-zero future payoff, because at any time t_j P_{t_j} can be expressed as the benefit, at death or maturity, or as the surrender value, multiplied for an indicator function that is equal to 1 only once, when the benefit or the surrender value is paid. The data used in the regression are given by the simulated discounted payoffs in all iterations (k) such that $T^{(k)} > t_n$, i.e. by

$$\sum_{j:\, t_n < t_j \le T^{(k)}} P^{(k)}_{t_j} v^{(k)} \left(t_n, t_j \right),$$

where $v^{(k)}$ denotes the simulated discount factor at the riskless rate in the k-th iteration. As happens for the corresponding random variable, $P_{t_j}^{(k)} \geq 0$ for any $j > n$ and there exists a unique j such that $P_{t_j}^{(k)} > 0$. Finally, we observe that in [LS01], with reference to a Bermudan-style put option, the authors recommend to limit the application of the regression step only to the simulated paths in which the option is in-the-money. Here, instead, we have to consider all paths because, first of all, we are valuing the whole contract and not simply the surrender option and, secondly, it is not possible to establish if this option is in-the-money since its underlying variable, the value of the residual contract, is not observable but could only be estimated by means of the same procedure under execution.

Step 4. The fair value of the whole contract at time 0, and hence the fair single premium, is given by

$$U = \frac{1}{H} \sum_{h=1}^{H} \sum_{j:\, t_0 < t_j \leq T^{(h)}} P_{t_j}^{(h)} v^{(h)} \left(t_0, t_j\right).$$

Remark: The premium for the corresponding European version of the contract can be simply computed as the average, over all iterations (h), of the simulated benefit paid at time $T^{(h)}$ discounted up to time t_0.

4 Tests of accuracy

The numerical accuracy of the method here proposed has been verified with reference to the valuation framework assumed in [Bac05]. This is a very simple framework, in which there is a single state-variable given by the value of the reference portfolio, that follows the binomial model by [CRR79]. However this is the only one in which we have exact analytic results to compare with those produced by the Monte Carlo approach: that is why we have chosen it to check the accuracy of this approach.

We recall that the contract analysed in [Bac05] is an equity-linked endowment policy. In particular, given a constant length Δ for each time interval $[t_{n-1}, t_n]$, we let $t_0 = 0$ and $t_n = n\Delta$, $n = 1, 2, ..., N$. Then, conditionally to the current level of the reference portfolio at time t_{n-1}, given by $F_{t_{n-1}}$, its level at time t_n can take only two possible values, respectively given by $F_{t_{n-1}}u$ and $F_{t_{n-1}}d$, with (risk-neutral) probabilities $q = \left(\exp(r\Delta) - d\right) / (u - d)$ and $1 - q = \left(u - \exp(r\Delta)\right) / (u - d)$, where r denotes the instantaneous riskless rate on an annual basis (deterministic and constant), $u = \exp(\sigma\sqrt{\Delta})$, $d = 1/u$, and σ represents the volatility of the reference portfolio, once again on an annual basis.

We have made a very large amount of numerical experiments, following different approaches to produce the simulated path of the reference portfolio, with different sets of parameters and by using both pseudo-random numbers and multi-dimensional low-discrepancy sequences. As basis functions we have employed either powers or Laguerre polynomials.

Observe that in each iteration (h), once the stochastic date T at which the benefit is due has been simulated, the simulated path of the reference portfolio can be generated

- *forwards*, from time t_1 to $T^{(h)}$, by using the conditional distribution above recalled;
- *backwards*, by simulating first the value of the reference portfolio at time $T^{(h)}$ from a binomial distribution, and after its values between times $T^{(h)}$ and t_1 from the corresponding conditional distributions. To this end we recall that, given $T^{(h)} = j\Delta$, $j = 1, 2, ..., N$, the possible values of $F_{T^{(h)}}$ are $F_0 u^i d^{j-i}$, $i = 0, 1, ..., j$, with (binomial) risk-neutral probability $\binom{j}{i} q^i (1-q)^{j-i}$. Given instead the level at time t_n of the reference portfolio, $F_{n\Delta} = F_0 u^i d^{n-i}$, $n = 2, 3, ..., N$ and $i = 0, 1, ..., n$, its level at time t_{n-1} can take only two possible values, that are $F_{n\Delta}/u$ and $F_{n\Delta}/d$, with probabilities i/n and $1 - i/n$. Note that if $i = 0$ or $i = n$ it actually takes only one value, given respectively by $F_{n\Delta}/d$ and $F_{n\Delta}/u$, and hence in these cases no simulations are required.

In particular, the accuracy of the results obtained by following these two different approaches has turned out to be

- the same, when pseudo-random numbers are employed,
- better backwards than forwards, when using low-discrepancy sequences (and for some sequences, e.g. the Halton one, very much better).

However, since Monte Carlo methods have a very low convergence speed, compared with quasi-Monte Carlo, and hence require a larger number of iterations in order to achieve the desired precision, it is better to use the backward approach also with them. In fact this approach does not require to keep track of all the entire simulated paths of the state-variable but only of its last simulated values, so that the spared allocated memory can be used in order to increase the number of iterations.

In Table 1 we show the results of some numerical examples. In all of them we have fixed $t_N = 10$ years, $\Delta = 1/12$, so that the length of each time interval is one month and the relevant dates are $t_n = n/12$, $n = 0, 1, 2, ..., 120$. The insured is assumed to be 40-years old at time 0, and the mortality probabilities used to simulate his(her) residual lifetime are extracted from the life table of the Italian Statistics for Females Mortality in 2001, with values corresponding to non-integer ages computed by linear interpolation. The initial value of the reference portfolio, F_0, is set equal to 1. Both the benefit paid at time t (death or maturity) and the surrender value R_t are assumed to be given by $\max\{F_t, F_0 \exp(gt)\}$, so that g represents an instantaneous minimum interest rate guaranteed. We have chosen a constant and a certain number of Laguerre polynomials as basis functions. This number has been fixed in such a way to maximize the accuracy of the results, within a maximum of 13. The number of iterations H, instead, has been fixed in such a way that the premium for the corresponding European version of the contract computed by means of the simplified valuation procedure described at the end of the previous section coincides with the exact premium analytically computed (at least until the 4-th decimal digit): in this way the error inborn in the Monte Carlo method is quite negligible and the residual error is mainly due to the regression. Finally, these results are obtained by using the multi-dimensional Weyl low-discrepancy sequence, that behaves very well although being the simplest one.

A few comments about our findings are in order. First of all note that, as expected, the number of iterations required to achieve the desired precision for the European version of the contract increases with the volatility of the reference portfolio. In particular, to produce the numerical results of Table 1 we have carried out $100,000$ iterations when the volatility parameter σ equals 0.15 or 0.25, and

Table 1. Numerical examples

r	g	σ	number of iterations H	number of Lag. pol.	estimated premium U	exact premium	error
0.03	0.00	0.15	100,000	9	1.0976	1.0976	0.0000
0.03	0.00	0.25	100,000	10	1.1971	1.1969	0.0002
0.03	0.00	0.35	1,000,000	13	1.2945	1.2956	−0.0011
0.03	0.02	0.15	100,000	11	1.1465	1.1464	0.0001
0.03	0.02	0.25	100,000	10	1.2604	1.2603	0.0001
0.03	0.02	0.35	1,000,000	13	1.3652	1.3674	−0.0022
0.05	0.00	0.15	100,000	9	1.0698	1.0700	−0.0002
0.05	0.00	0.25	100,000	9	1.1555	1.1555	0.0000
0.05	0.00	0.35	1,000,000	13	1.2444	1.2449	−0.0005
0.05	0.02	0.15	100,000	9	1.0975	1.0977	−0.0002
0.05	0.02	0.25	100,000	8	1.1964	1.1972	−0.0008
0.05	0.02	0.35	1,000,000	13	1.2938	1.2955	−0.0017
0.05	0.04	0.15	100,000	7	1.1458	1.1460	−0.0002
0.05	0.04	0.25	100,000	7	1.2599	1.2598	0.0001
0.05	0.04	0.35	1,000,000	13	1.3635	1.3673	−0.0038

$1,000,000$ iterations when $\sigma = 0.35$. The average absolute error is equal to 18.6 basis points (bp) when the volatility is high, to 2.4 bp when it is medium and only to 1.4 bp when it is low. Moreover, the number of Laguerre polynomials required to optimize the regression procedure is equal to 9 (on the average) when the volatility parameter σ equals 0.15 or 0.25, and always to the maximum number here fixed, 13, when $\sigma = 0.35$. This indicates that, in case of high volatility, it would be better to further increase the number of iterations and/or the number of basis functions. Summing up, we can conclude that the results obtained are in general very good, so that the methodological approach here proposed seems to be suitable for application to more realistic valuation frameworks, with stochastic interest rates, stochastic volatility, jumps, stochastic mortality trends, ...

5 Summary and conclusions

In this paper we have proposed a method to extend the Longstaff-Schwartz Least Squares Monte Carlo approach to the case of life insurance contracts embedding a surrender option. Then we have applied it to a very simple framework, in which there are exact analytic solutions, in order to verify its accuracy. The accuracy tests indicate that this method performs very well, especially for low or medium-volatile reference portfolios, and hence the next step is to apply it to more sophisticated frameworks, even if there will be no way to compare the results obtained with exact ones.

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